MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 97 No. 6

MEETING OF 1937 APRIL 9

Dr. H. Spencer Jones, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Mark Bernard Brewer Heath, Urania, Firleigh Road, Kingsteignton, Devon ;
Ernest George Martin, Royal Observatory, Greenwich, S.E. 10 ;
Arnold Smith, M.A., Maltby Grammar School, Rotherham, Yorks ; and
Miss Maureen Elizabeth Tindall, 21 Upper Park Road, Hampstead, N.W. 3.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Dr. Beno Gutenberg, Seismological Laboratory, 220 San Rafael Avenue N., Pasadena, Cal., U.S.A. (proposed by Harold Jeffreys) ; and
Dr. József Wodetzky, Professor of Astronomy, Budapest University, Múzeum-Krt. 6/8, Budapest VIII (proposed by A. S. Eddington).

Eighty-one presents were announced as having been received since the last meeting, including :—

Arthur R. Hinks, Astronomy. Second Edition. Revised and largely Rewritten (presented by the Author) ; and
Mary Proctor and A. C. D. Crommelin, Comets, their Nature, Origin, and Place in the Science of Astronomy (presented by the Publishers).

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The President announced that the following Associates had been elected by the Council :—

Dr. Bernard Lyot, Meudon Observatory ;
Professor J. H. Oort, Leiden Observatory ; and
Professor Gregory A. Shajn, Simeis Observatory.

The President further announced that a joint meeting with the Royal Society would be held on May 27, at the rooms of the Royal Society in Burlington House, to discuss the results of the eclipse of 1936 June 19.

PHOTOMETRY OF THE ANDROMEDA NEBULA, M 31.

R. O. Redman and E. G. Shirley.

Summary.—The nebula has been traced to a major axis diameter 267′ ; minor 89′. The axis ratio, for points of equal surface brightness, varies from near 1 at the centre down to 0·31 ; in the outermost region measured it is 0·33. Comparison is made with the photoelectric measures of Stebbins and Whitford. The total luminosity is $3^{m}.6 \pm 0^{m}.3$ (photographic), while the surface brightness at the centre is at least $14^{m}.8$/(second of arc)^2. At the centre of M 33 the surface brightness is $20^{m}.3$/(second of arc)^2.

Some months ago we were asked by Mr. J. H. Reynolds to make comparative measures of the surface brightness near the centres of M 31 and M 33. The work on the first of these, the Andromeda spiral, has grown beyond the scope originally intended, and the results described here will be found to relate mostly to it.

For the central regions of each nebula, photographs were taken with the 3-foot Common reflector at the Solar Physics Observatory, Cambridge, the calibration and measurement following the procedure described in *M.N.*, **96**, 588, 1936 (Method 2). For M 31 it was thought that it would be worth while to extend this work into the outer parts of the nebula, using small scale photographs, taken with an $f/3$ Aldis lens of 12 inches focal length. The following description applies to this nebula only, unless otherwise stated.

The distribution of surface brightness was measured along major and minor axes, the results from different plates being fitted together with the aid of log I graphs. The large scale plates, 14 exposures in all, were used to about 20′ from the centre along the major axis and 10′ along the minor. Two small scale photographs were used from this point outwards, there being sufficient overlap for a satisfactory fit to be made. The Aldis lens gives satisfactory star images with normal exposure, but with over-exposure it can be seen that each image is surrounded by a faint halo, which in extreme cases may be more than 10′ in diameter and is probably due to imperfect chromatic

TABLE

Distribution of Luminosity along the Axes of M 31

The intensity unit is such that log I = 2 corresponds to a surface brightness $16^{m}\cdot2$/(second of arc)^2.

Major Axis (S.W. to N.E.)		Minor Axis (S.E. to N.W.)	
Distance from Centre	Log *I*	Distance from Centre	Log *I*
′		′	
125	$\bar{3}\cdot62$	44	$\bar{3}\cdot77$
100	$\bar{2}\cdot41$	40	$\bar{2}\cdot10$
90	·78	30	·76
80	·94	20	$\bar{1}\cdot31$
70	$\bar{1}\cdot08$	10	·88
60	·23	5	0·24
50	·34	4	·34
40	·47	3	·52
30	·61	2	·77
20	·84	1	1·13
10	0·15	0·5	1·38
5	·43	0	2·00
4	·55	0·5	1·34
3	·72	1	1·12
2	·92	2	0·74
1	1·20	3	·45
0·5	·37	4	·26
0	2·00	5	·10
0·5	1·39	10	$\bar{1}\cdot68$
1	·21	17	·48
2	0·92		
3	·73		
4	·58		
5	·45		
10	·15		
20	$\bar{1}\cdot90$		
30	·73		
40	·63		
50	·54		
60	·34		
70	·20		
80	·05		
90	$\bar{2}\cdot88$		
100	·66		
120	·30		
142	$\bar{3}\cdot63$		

correction.* However, owing to the great size of the nebula and to the small intensity gradient in the outer regions, this imperfect definition should lead to no important distortion of the measured light distribution. The natural

* In fairness to the lens it should be stated that it was not designed for this class of work. Nevertheless, while the photographs taken with it are very useful, especially in this case, where the nebula is much bigger than the field of the 3-foot reflector, we wish to point out that the accuracy given by them is in general considerably lower than that attained with the reflector negatives.

difficulty of working on a small scale is somewhat increased, since star images, as well as the image of N.G.C. 205, which is situated very near the minor axis, are correspondingly large relative to the main nebula and more difficult to avoid in the photometry. Towards N.G.C. 205 the distribution along the minor axis could be measured only 17′ from the centre, and to obtain the total luminosity we were obliged to assume that from this point outwards there is symmetry with the opposite side. In actual fact there is a certain amount of asymmetry on both axes, some of which is apparent on visual inspection of most photographs, but it is not very great and should not introduce any large error into the total luminosity. Measurement of the distribution on one of our plates at 45° to the axes is consistent with our assumption that the isophotal contours are approximately elliptical.

On the major axis, measurements have been made to a diameter 267′, on the minor axis to a radius 44½′ in one direction and 17′ in the other (see Table). With the assumption of minor axis symmetry beyond 17′ radius, the axis ratio of the isophotal contours varies from about 0·9 near the centre to 0·3 at the edge (fig. 1). The tendency to circularity at the centre seems to be real, although at radii less than 0′·2 it could be instrumental, and seems to be an indication that the very bright nuclear regions are globular. According to Shapley,* the major axis extends to 270′ and the minor to 240′ diameter. We think it open to question whether the second of these estimates should be accepted, although the first coincides very closely with our own. There is great difficulty with small scale photographs, taken with fast lenses and heavily over-exposed, as these must be, of separating on microphotometer tracings the very faint parts of the nebula from small clumps of stars, since probably every star image has around it a faint halo of the kind mentioned above. Assuming the lens gives perfect definition, our own tracings could be interpreted as showing extension to about 90′ radius on the side towards N.G.C. 205 and about 110′ on the other. But it seems to us better, in view of the known imperfection of the objective, to adhere to the more conservative estimates given at the head of this paragraph. In any case, even if the minor axis is ultimately traced to a diameter of the order of 4°, on large scale photographs or by means of photoelectric observations, it seems extremely unlikely that the structure revealed will be at all regular, or of the nature of an essentially spherical envelope.† The dubious extensions on our minor axis tracings seem on the one side to be associated, optically or physically, with N.G.C. 205, while on the other side there is an intermediate break where the surface brightness diminishes to something very near indeed to the sky background. The character of the *major axis* tracings is in this respect quite distinct, for they show a more or less continuous fall of brightness from centre to edge.

Stebbins and Whitford,‡ whose photoelectric observations were the first

* *M.N.*, **94**, 804, 1934.

† Our conclusions are substantially those of Vocca (*Contributi della R. Specola di Brera-Milano*, No. 29, 1935), who rightly points out that the only satisfactory method of investigating this matter is to map the surface brightness distribution of the whole nebula.

‡ *Proc. Nat. Acad. Sci.*, **20**, 93, 1934.

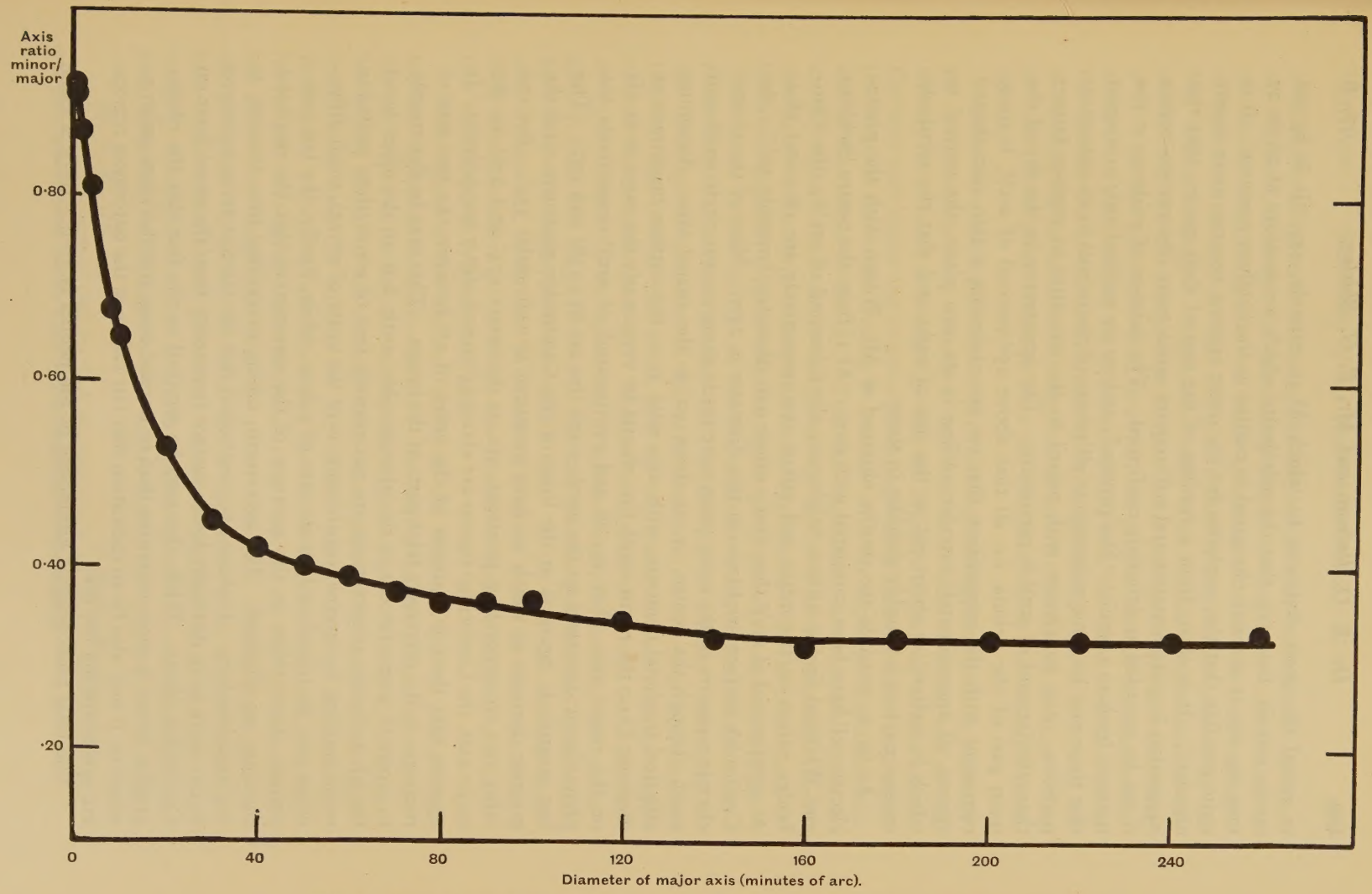

FIG. 1.—*Ratio of axes of isophotal contours, assuming symmetry on the minor axis beyond radius 17′.*

to reveal the great distances to which M 31 extends, say "it is by no means certain, however, that the nebulosity which we measure at 26 or 27 mag./sq. sec. is as much elongated in outline as the brighter portions. It is quite possible that the isophotes for the outer regions become more nearly circular". It is clear, from a reading of the rest of their paper, that this suggestion is quite tentative and will require much more observation before it can be regarded as definitely confirmed. The balance of evidence at the moment leads to a verdict "Not proven", and we are entitled only to suspect that there may be some reduction of ellipticity of the nebula in its outermost portions. Any conclusion with regard to the structure in regions fainter than 26^m/(second of arc)^2 is premature. The appearance in the sky of the main part of the nebula, *i.e.* all that above 25^m/(second of arc)^2, is quite consistent with the supposition that we are observing a thin disc-shaped system, of approximately circular outline in its own plane, the normal to which is inclined at about 70° to the line of sight and that the relatively intense nucleus is roughly globular in shape.

As far as possible the results obtained at Mt. Wilson with the photoelectric cell have been compared with ours. At 1′ from the centre Stebbins and Whitford find a surface brightness $18^m.6$/(second of arc)^2; the Cambridge values on the major and minor axis respectively are 18·2 and 18·4. At 25^m/(second of arc)^2 they find a minor axis diameter "roughly 50′"; the Cambridge surface brightness at this diameter is 23·7. Most of the photoelectric measures in the outer parts were made along a great circle north and south through the centre, *i.e.* at about 40° to the major axis. Assuming elliptical isophotal contours, with axis ratio 3 : 1, the surface brightness at diameter D on the north-south line should be very nearly the same as at 2D on the major axis. For 25, 26 and 27^m/(second of arc)^2 respectively the photoelectric diameters on the north-south line are 87′, 138′ and 177′. The last magnitude figure is at the limit of the Cambridge measures, and the greatest diameter at which we have measured is well under 354′. For the other two corresponding positions, viz. at diameters 174′ and 276′ on the major axis, the Cambridge figures are about $24^m.0$ and $26^m.7$ respectively. It appears that there is a scatter of the order of 1^m between the two sets of measures in the extremely faint parts of the image. This may be due mostly to errors of measurement in the photographic work, but on the other hand the photoelectric observations are not entirely free of error (their probable error, judging by internal consistency, is of the order of $0^m.25$), small differences may be introduced on account of colour, while, finally, the last comparison above relies on the accuracy of the assumption that the isophotal contours are elliptical. The agreement, taking everything into account, is not unsatisfactory. It should be mentioned that the faintest areas measured in our work have distinctly lower surface luminosity than the usual limit on Cambridge plates. While this may be ascribed to the fact that the photographic image is more extensive than in normal cases and therefore easier to measure, it may also be an indication that our values in the outermost regions are systematically too low.

The total photographic luminosity of the nebula is $3^m.6 \pm 0^m.3$, a value

obtained by integration, following the assumption that the isophotal contours are elliptical, although not necessarily similar or concentric. Reduction to the stellar magnitude scale was made by means of one extra-focal exposure of Selected Area 20, four stars being used and the magnitudes taken from the Mt. Wilson Catalogue. Experience with other similar determinations has shown that the star comparison in itself has a probable error not exceeding $0^m.1$. In this particular case, however, the greater part of the uncertainty arises from other causes, viz. (*a*) in smoothing over the spiral structure, (*b*) in the assumption of the distribution in the neighbourhood of N.G.C. 205, (*c*) in deciding on the small scale plates exactly how far the nebula extends. The probable error given seems to be a fair estimate, taking into account these factors.

It has been realized in the last few years that extragalactic nebulæ generally are considerably more extensive than had once been thought, and it has already been found at Cambridge that the brighter ellipsoidal nebulæ are for the most part bigger and brighter photographically than published lists state, the error on the whole tending to increase with the apparent brightness and size of the nebula in the sky. That the magnitude just quoted for M 31 is much brighter than earlier estimates, *e.g.* the $5^m \pm$ of Shapley and Ames's catalogue, is thus quite in line with recent experience. On the other hand, it is also substantially brighter than the figure, $4^m.5$, quoted recently by Hubble.* This seems to rest chiefly on the photoelectric work of Whitford,† which refers only to an area of diameter 128′, centred on the nebula. According to our distribution measures an appreciable amount of the total light comes from an area outside this circle, and the correction on that account might amount to as much as $0^m.3$. Whitford's original $4^m.31$ was corrected by $0^m.13$ on account of the difference in colour sensitivity between the photographic plate and his photoelectric cell, and it seems possible that some small additional uncertainty was introduced here. If we take into account both sets of measures, all that can be said at the moment is that the total luminosity is in round numbers 4^m, although the present writers are inclined to believe that the nebula is brighter than this.

With regard to the nucleus, as far as our measures can be pushed the surface brightness increases steadily to the centre, the rate of increase becoming very rapid in the last few tenths of a minute of arc. Mr. E. G. Williams has kindly placed at our disposal some measurements he made of the light distribution near the nucleus on Mt. Wilson plates taken by Dr. Hubble. Between 0′·1 and 0′·2 from the centre the Mt. Wilson and Cambridge figures for the two axes agree, with residuals in log I up to 0·05 (fig. 2), but at radii less than 0′·07 the greater resolving power of the Mt. Wilson plates gives a sharper peak. Even so, the latter fail at the centre itself to give anything more than a distribution practically identical with that of an ordinary star image.‡ Fitting the two sets of figures together as well as possible, the central peak, according to Williams, is $1^m.4$ higher than in our measures. (Owing to

* *Ap. J.*, 84, 168, 1936. † *Ap. J.*, 83, 424, 1936.

‡ It should be noticed that the limit of resolution is set by the photographic plate and microphotometer rather than by the telescope or the seeing.

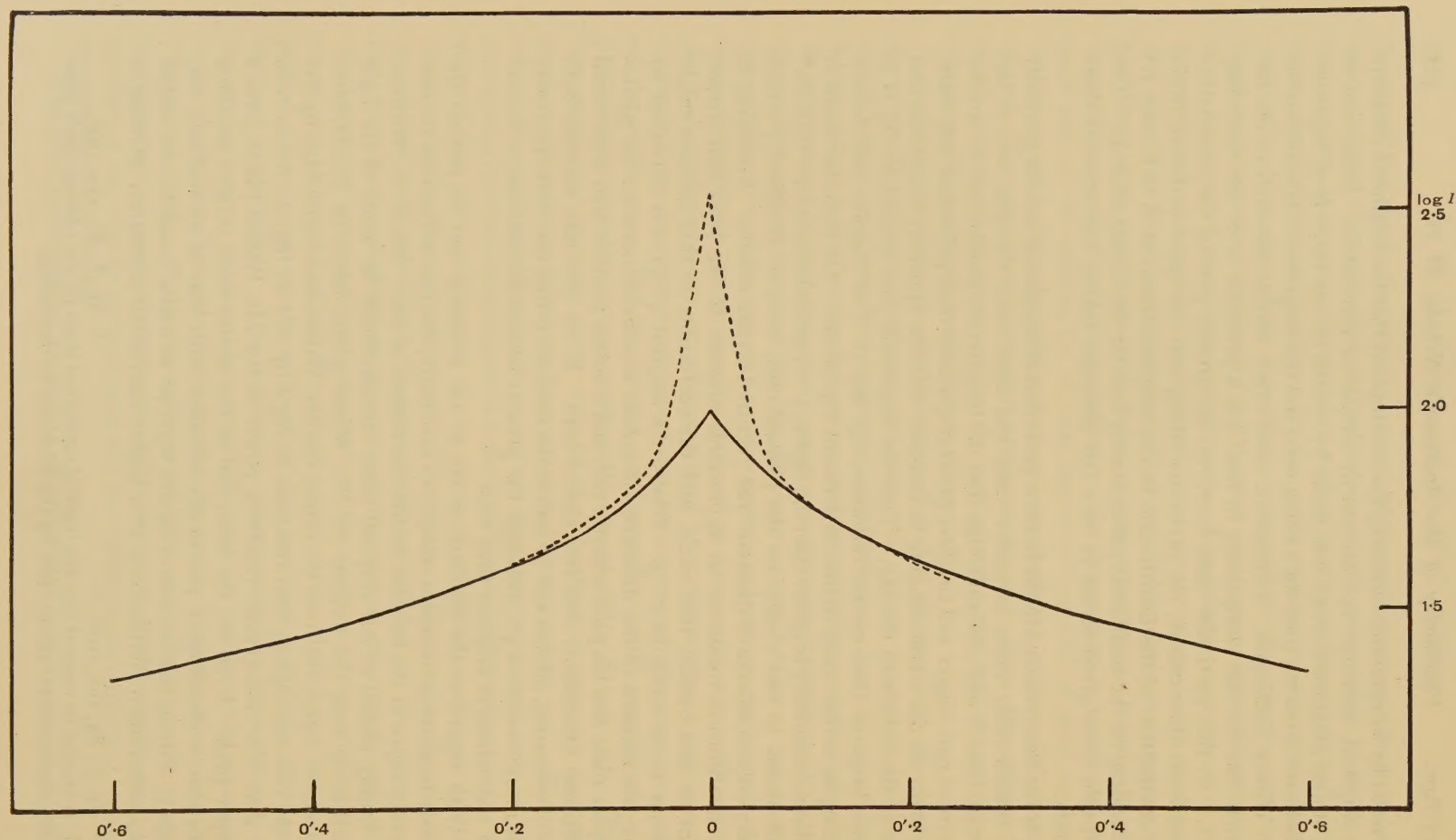

FIG. 2.—*Luminosity distribution along the major axis near the centre of M* 31.

——— *Cambridge measures.*

..... *Williams's measures* (*Mt. Wilson plates*).

the extremely small area involved, this has no appreciable effect whatever on the total luminosity of the nebula.) Reduced with the aid of the other Cambridge observations, the Mt. Wilson surface brightness at the centre is then $14^{m}.8$/(second of arc)^2. On account of finite resolving power this is a minimum value, representing the least concentration which must really occur at the centre. We understand that unpublished Mt. Wilson material shows that the spectrum of the nucleus is of a peculiar dwarf A type. It seems likely that this is mainly produced by dwarf A stars, and if the average absolute magnitude of these is +1, the star density at the centre attains at least 1000/(parsec)^3.* There is, however, a very steep fall of density outwards, and at $0'.1$, *i.e.* about 7 parsecs from the centre, where the surface brightness is $17^{m}.2$/(second of arc)^2, the density cannot well exceed 100 stars/(parsec)^3.

The centre of M 33 is quite different in character from that of M 31. There is no gradual increase of luminosity to a quasi-stellar nucleus. The spiral arms spring from a hazy mass which is of roughly constant surface brightness and has a diameter about $1'.5$. Lying along the spiral arms there are large numbers of stellar or quasi-stellar objects, apparently belonging to the nebula, and some also are to be found in the nebulous centre just mentioned. A few of the stars in the immediate neighbourhood of the centre may be foreground objects. The Cambridge reflector is unable to reveal any detail which can be definitely assigned as the nucleus of the nebula and which at the same time has enough area for the surface brightness to be measured. Excluding stellar objects, the maximum surface brightness of the "hazy mass", measured with one standard star exposure (Selected Area 46), is about $20^{m}.3$/(second of arc)^2.

DYNAMICAL EFFECTS OF RADIATION IN THE SOLAR SYSTEM.

H. P. Robertson.

(Communicated by Professor H. N. Russell)

1. *Introduction.*—In connection with his investigations on radiation pressure, J. H. Poynting † at one time considered the consequences of absorption and re-emission of solar radiation for the motion of small bodies. Of these, perhaps the most important for cosmogony was a tangential drag, which would give rise to a decrease in the angular momentum of the particle, and thus cause it eventually to fall into the Sun. This drag Poynting attributed to a back-pressure of radiation tending to retard the motion through the ether of the emitting body, which he conceived as due to a

* Sinclair Smith (*Ap. J.*, **82**, 192, 1935) has deduced $8.8\ 10^5$ stars/(parsec)^3 for the centre of the companion nebula, N.G.C. 221.

† *Phil. Trans. Roy. Soc.*, A, **202**, 525, 1903; reprinted, with correction of numerical slips, as Art. 20, p. 304, of his *Collected Scientific Papers*, Cambridge, 1920. Reference may also be made to these collected works for a number of Poynting's addresses dealing with the same subject, notably Arts. 22, 64–67.

crowding-up of radiation in front of the particle and to a corresponding thinning-out behind; the magnitude of the resulting force (as corrected in a note at the end of the paper referred to above) was given as $\frac{4}{3}Rv/c^2$, where R is the rate at which the particle is radiating energy, v is its velocity through the ether, and c is the velocity of light. Now, since the Sun's gravitational effect on bodies of equal density goes down with the cube of their linear dimensions, whereas the rate at which they receive and re-radiate energy decreases only with the square of these dimensions, the effect in question should be the more marked the smaller the body. Thus, according to Poynting, a spherical body of radius 1 cm. and density 5·5, starting from the Earth's distance from the Sun and with the Earth's orbital velocity, could make at most some 10^8 revolutions before falling into the Sun, while one of radius 10^{-3} cm. could survive at most some 10^5 revolutions.

Although Poynting emphasized the cosmogonic implications of these results, they do not seem to have been widely recognized at the time.* The subject was reopened some years later by J. Larmor, in an address before the International Congress of Mathematicians at Cambridge, in which he gave an alternative approximate treatment, on classical electromagnetic theory, of "the retarding force exerted on a body translated through the æther with uniform velocity [v], arising from its own radiation," and showed that, to terms of first order in v, the force in question was given by Rv/c^2 †—three times Poynting's corrected value.

But that this force could by itself act as a brake on the movement of the particle is clearly in contradiction with the theory of relativity,‡ which is in turn consistent with the electromagnetic theory on which the result was derived. This circumstance led L. Page to undertake a detailed examination of the matter, and he was indeed able to show that, to terms of the order in question, the velocity of a moving body does *not* suffer a retardation in consequence of its own radiation alone.§ That this is in fact the case was subsequently admitted by Larmor in a Postscript to Poynting's *Collected Papers*, where he states (p. 757) that "the remarkable result seems to be established that an isolated body cooling in the depths of space would not change its velocity through the æther, the retardation due to the back thrust of radiation issuing from it being just compensated by increase of velocity due to momentum conserved with diminishing mass." And here, for the first time, the true sufficient cause of the drag in question was clearly stated, and the paradox resolved; in Larmor's words, "But for Poynting's particle

* An interesting application of these notions is to be found, however, in the attempt of H. C. Plummer, *M.N.*, **65**, 229, 1905; **67**, 63, 1906, to explain in this way the anomaly in the mean motion of Encke's comet.

† *Proc. 5th Internat. Cong. Math.*, **1**, 197, Cambridge, 1913; reprinted in part in Poynting's *Collected Papers*, p. 426. The result quoted above is that obtained on p. 215 of the *Proceedings*.

‡ As was pointed out in *The Observatory*, **40**, 278, 1917, in a review of some related work, to which Larmor replied in a letter to *Nature*, **99**, 404, 1917, insisting on the reality of the retardation.

§ *Phys. Rev.*, **11**, 376, 1918; **12**, 371, 1918; summary of first article in *Proc. Nat. Acad. Sci.*, **4**, 47, 1918.

describing a planetary orbit the radiation from the Sun comes in, which restores the energy lost by radiation from the particle, and so establishes again the retarding force $[-Rv/c^2]$." However, Larmor goes further and asserts that the particle is subject to yet another drag, of the same amount, due to the "astronomical aberration of light," thus doubling the result which he had previously obtained. Closer examination shows, however, that this doubling—which, in its entirety, would again lead into contradiction with the theory of relativity—affects only the radial component of the drag, to terms of the order retained by Larmor, and is therefore ineffective in producing a further decrease of angular momentum.

At the instance of Professor H. N. Russell, who called my attention to Poynting's paper, I have recently undertaken an examination of the whole question from the standpoint of the theory of relativity. On communicating my findings to him, he has urged that I make them public ; this, then, is the purpose of the present note. In the following Section the exact relativistic equations are derived, and in subsequent Sections are applied (to terms of first order in v) to those questions which first attracted Poynting's attention. I may say that for an understanding of these latter applications, the reader unfamiliar with the notation and methods of the theory of relativity need not have read the more rigorous derivation of Section 2, for to that approximation, in which only linear terms are preserved, the equations of motion are readily interpretable in classical terms, as set forth at the opening of Section 3. Aside from its intrinsic interest, the precise relativistic treatment was deemed necessary to establish beyond all cavil the existence of just those linear terms, and no others, which are employed in the subsequent development. It is a pleasure to record here my indebtedness to Professor Russell for his stimulating comments and advice, particularly on the astronomical aspects of the subject, to which the latter Sections are devoted.

2. *Relativistic Equations of Motion.*—We propose, then, to investigate the motion of a small, intrinsically spherical body in a beam of radiation, under the supposition that it absorbs all the energy falling upon it, and that it re-emits this energy at the same rate isotropically with respect to a reference system in which it is instantaneously at rest. This latter circumstance suggests that we first set up the equations in the rest system, and then transform them back into the original inertial system ; this procedure will have the additional advantage that we may take over, without essential modification, results on radiation pressure which have been established by purely classical methods for bodies at rest. As will be seen in the sequel, it will suffice to consider the problem in the frame of the special theory of relativity, for the modifications necessitated by the general theory can most easily be made later, in case they are desired.

Let the original reference system S, which may in the applications be taken as one in which the Sun is instantaneously at rest, be defined by a Minkowski co-ordinate system $x^\mu = (t, x, y, z)$, in which the invariant line element assumes the canonical form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - (dx^2 + dy^2 + dz^2)/c^2, \qquad (2.1)$$

where the repeated indices μ, ν are summed over their range 0, 1, 2, 3. The rotational motion of the body is here ignored, as its energetic effects are presumably negligible and we are assuming that in all cases the mechanism of transfer of energy within the body is such that it is ultimately radiated isotropically from all portions of the surface of the body; under these conditions the motion of the body may be specified by a unit four-vector of velocity u^μ, satisfying the equation

$$g_{\mu\nu}u^\mu u^\nu = 1, \tag{2.2}$$

or by its energy-momentum vector mu^μ, where m is its proper mass. For considerations not involving diffraction or selective pressure—the influence of which is touched upon later—the primary radiation may be thought of as a plane-parallel beam, defined in direction by a null-vector l^μ whose components are chosen subject to the convenient normalization condition $l^0 = 1$ in the given reference system S:

$$g_{\mu\nu}l^\mu l^\nu = 0, \qquad l^0 = 1; \tag{2.3}$$

the density of energy in this incoming beam may then be taken as the temporal average d of the electromagnetic energy density. Under these circumstances the primary radiation is completely specified by the energy-momentum-stress tensor

$$e^{\mu\nu} = dl^\mu l^\nu, \tag{2.4}$$

representing a beam of energy density $e^{00} = d$ carrying momentum at the rate d per unit cross-section in the spatial direction defined by the components l^α ($\alpha = 1, 2, 3$) of l^μ, and exerting a pressure d in this direction.

In order to carry out the programme outlined above, we introduce, in the neighbourhood of an event E in the history of the particle, a rest system Σ_E, with co-ordinates $\xi^\mu = (\tau, \xi, \eta, \zeta)$. We denote the components in Σ_E of the vectors u^μ, l^μ, defining the motion of the particle and the direction of the beam, by the corresponding Greek letters v^μ, λ^μ. Then at E

$$v^\mu = \delta_0{}^\mu, \qquad \lambda^\alpha = c\lambda^0 \nu^\alpha \qquad (\alpha = 1, 2, 3), \tag{2.5}$$

where the ν^α, defining the direction of the beam in the proper-space of the particle, may, in consequence of the fact that λ^μ is a null-vector, be considered as the three components of a unit vector in the sense of ordinary vector analysis; it is to be noted that we are here not free to choose λ^0 unity, for the normalization is completely given by the condition that the corresponding quantity l^0 in the reference system S be unity. The energy-momentum vector of the particle is here given by mv^μ, and the energy-momentum-stress tensor of the primary radiation by

$$\epsilon^{\mu\nu} = \delta\lambda^\mu\lambda^\nu/\lambda_0{}^2, \tag{2·6}$$

where δ is its energy density ϵ^{00} in the rest system Σ.

The assumption that the particle re-radiates energy at the same rate at which it receives it insures that the proper mass m of the particle is a constant of the motion, and this and the fact that v^μ is a unit vector together imply that the time rate of change of mv^0 vanishes at E. Now the proper rate of change of the spatial components mv^α of the energy-momentum vector is equal

to the impressed force, and since the particle is assumed to possess spherical symmetry and to radiate isotropically in the rest system Σ, this force can only act in the direction ν^α of the spatial projection of the null-vector λ^μ of the incoming beam. Since the particle is instantaneously at rest in the system Σ the classical results on radiation pressure may be called upon for the determination of the magnitude ϕ of the force in question ; we are at the moment primarily interested in the fact that it is in any case directly proportional to the energy density δ of the incoming beam—even where diffraction and selective pressure play an important rôle, although we may then be compelled to interpret much of what has gone before in a statistical sense. We may therefore write

$$\phi = A\delta, \tag{2·7}$$

where in the case of principal interest below, in which the radius a of the sphere is large compared with the wave-length of the radiation, the factor A is merely the projection πa^2 normal to the beam of the area of the illuminated hemisphere ; we prefer, however, to retain the more non-committal formulation (2.7) to allow for cases in which diffraction or resonance may be thought of as giving rise to a larger *effective cross-section* A.*

The equations of motion thus obtained in the rest system Σ may now be brought together in the four-dimensional formulation

$$\frac{dmv^\mu}{d\tau} = \frac{\phi}{c\lambda^0}\{\lambda^\mu - \lambda^0 v^\mu\} ; \tag{2.8}$$

for, on taking into account the relations (2.5), the temporal component $\mu = 0$ of the right-hand side is seen to vanish at E, thus expressing the conservation of mass, and the spatial component $\mu = \alpha$ is equal to the component $\phi\nu^\alpha$ of the force due to radiation pressure. The first term $\phi\lambda^\mu/c\lambda^0$ represents the total power-force four-vector due to the incident radiation, with which it coincides completely in direction, and the second that due to the radiation re-emitted at the rate ϕc.†

* Investigations on radiation pressure when diffraction is of importance are to be found in papers by K. Schwarzschild, *Sitzungsber. Bayer. Akad. d. Wiss.*, **31**, 293, 1901, and by P. Debye, *Ann. d. Phys.*, **30**, 57, 1909. The selective pressure effects due to resonance have also been treated by Debye, *loc. cit.*, by L. Page, *Astrophys. Journ.*, **52**, 65, 1920, by W. Baade and W. Pauli, *Naturwiss.*, **15**, 49, 1927, and by A. Unsöld, *Zeits. f. Phys.*, **44**, 793, 1927. The application of this latter effect to the form of the tail of a comet, together with references to the literature, have been given recently by K. Wurm, *Zeits. f. Astrophys.*, **10**, 285, 1935—where it is pointed out that the pressure on CO^+ due to the solar radiation is of the order of 100 times the gravitational attraction.

† It may be remarked, however, that the validity of these equations is not fully circumscribed by the particular scattering mechanism here contemplated. For if, for example, we consider the effect of the radiation on a small perfectly reflecting sphere, the equations of motion are exactly the same as those above for complete absorption and subsequent re-emission, the additional force on that region of the sphere in which the normal makes an angle of less than $\pi/4$ with the direction from which the beam is coming being just counterbalanced by an equal and opposite force on those portions in which the angle is between $\pi/4$ and $\pi/2$, as can be shown from Poynting's work on normal pressure and tangential stress due to radiation, *C.S.P.*, p. 318.

In order to obtain from (2.8) the equations of motion of the particle in terms of the original system S with co-ordinates x^μ, it is only necessary to remark that the λ^μ, v^μ appearing in (2.8) are the components of the two four-vectors which are represented in S by l^μ, u^μ, and to express in scalar form the quantities λ^0, ϕ occurring in the coefficients. With regard to λ^0, we note that in the rest system Σ it is the scalar product of λ^μ with v^μ, and is therefore to be replaced by the scalar

$$w = l_\mu u^\mu (= \lambda^0). \tag{2.9}$$

Next, we note that δ, which appears in the expression (2.7) for the proper-force ϕ, is simply the component ϵ^{00} of the energy tensor (2.6) in the rest system Σ, and may therefore be replaced by the double inner product $\epsilon^{\mu\nu} v_\mu v_\nu$ of this tensor with the velocity vector v^μ. But the value $e^{\mu\nu} u_\mu u_\nu$ of this scalar in terms of the system S is, by (2.4) and (2.9), simply $w^2 d$, and we may therefore write

$$\delta = w^2 d, \quad \text{or} \quad \phi = w^2 f, \tag{2.10}$$

where we have introduced for convenience the force $f = Ad$ which would act on the particle if it were at rest in the system S.

The equations of motion of the particle in the original reference system S are therefore

$$\frac{dmu^\mu}{ds} = \frac{fw}{c}\{l^\mu - wu^\mu\}, \tag{2.11}$$

where s is the proper time of the particle and, in recapitulation, $w = l_\mu u^\mu$, $l^0 = 1$, and f is the force which *would* act on the particle if it *were* at rest in the system S—and is carefully to be distinguished from the force which *does* act on it when viewed from the system Σ in which it *is* at rest.

The above development has been based on the special theory of relativity, and we have here to add a few words concerning the modifications required on going over to the general theory. If we consider the particle as a test particle, whose effect on the gravitational metric due to the Sun or other bodies is negligible, and ignore possible effects due to the interaction of radiation and gravitation in computing the impressed force f in normal co-ordinates at E, the only change required is the addition of the inertial terms

$$m\left\{ \begin{matrix} \mu \\ \nu\sigma \end{matrix} \right\} u^\nu u^\sigma \tag{2.12}$$

on the left-hand side of the equations of motion (2.11), where the Christoffel symbols are computed from the given metric. This is equivalent to adding, in the classical approximation, the Newtonian gravitational force, as is done directly in the succeeding section, and, in the next approximation, the small relativistic correction which is responsible for the perihelion advance of a Keplerian orbit.

Before proceeding to the applications, let us examine the equations (2.11) with reference to the point which has caused so much confusion in the literature referred to in the introductory section. First, as the equations stand they must, of course, lead to the conservation of proper mass—as can

be seen by multiplying through by u_μ and summing with respect to μ, taking into account the relations (2.2) and (2.9). Clearly the second term does here represent an actual retarding force, which must be combined with the force in the direction l^μ due to the radiation pressure of the incoming beam. But equally clearly this force would not give rise to a retardation of the velocity if the first term were left out of account, for we would then have

$$m\frac{du^\mu}{ds}+\frac{dm}{ds}u^\mu=-\frac{\phi}{c}u^\mu, \tag{2.13}$$

and since in this case the particle would be losing mass at the rate $\phi c/c^2$ at which energy is being radiated away (in mass units), the second term on the left just cancels that on the right, whence the velocity u^μ is constant. This is in fact the exact relativistic form of the explanation proposed by Larmor in the approximate treatment referred to in Section 1, and is in full agreement with the more detailed studies by Page. The explicit introduction of the radiation force, which represents the immediate source of the energy emitted, serves to remove the compensating term on the left in the above equation, and to restore the drag as giving rise to a true retardation of velocity. On the other hand, it is apparent that any effect attributable to aberration must already be included in the above treatment; its introduction can at most suggest an alternative interpretation of the results, and cannot lead to additional terms, as was held by Larmor.*

3. *Newtonian Approximation. Equations of Motion of Particle in Field of the Sun.*—For problems in which the specifically relativistic effects may be expected to play no important rôle, the equations derived in Section 2 above may be reduced to a simpler form, in which they are directly comparable with the approximate equations derived by Poynting and by Larmor. Here we are interested only in terms of first order in the ratio of the velocity of the particle to that of light, and we may accordingly revert to the Newtonian scheme, expressing the equations of motion in terms of the two vectors

$$v^\alpha=\frac{dx^\alpha}{dt}, \qquad n^\alpha=l^\alpha/c, \tag{3.1}$$

the second of which is the unit vector in the direction of the incident beam. The various quantities involved in the equations (2.11) may then be taken, to

* It may appear paradoxical that there can be a retardation transverse to the beam in one reference system and not in another, but it is to be remembered that the angle between the spatial projections of two four-vectors, such as l^μ and u^μ, is not invariant. Thus in the case in which the incoming beam is rigorously plane-parallel, the motion of the particle, as viewed from the standpoint of an inertial system Σ with respect to which it was at one time at rest, is always along the direction of the spatial projection l^α of the null-vector defining the beam, whereas from the standpoint of an arbitrary reference system S there will in general be a transverse component. The direct integration of the equations (2.11) for this case shows that the ratios of the velocity components u^μ approach those of the null-vector l^μ as $s\to\infty$, and that on following the motion backward in time these ratios approach those of the components k^μ of some other null-vector; the possible rest systems Σ are characterized geometrically by the fact that their time axes lie in the plane determined by the two null-vectors k^μ, l^μ.

terms of first order in v^α, as

$$u^0 = 1, \qquad u^\alpha = v^\alpha, \qquad w = 1 - v_n/c, \tag{3.2}$$

where v_n is the component of the velocity v^α in the direction n^α; the equations of motion then become, to this order,

$$m\frac{dv^\alpha}{dt} = f(1 - v_n/c)n^\alpha - fv^\alpha/c. \tag{3.3}$$

The first term on the right may be interpreted as that due directly to the radiation pressure, in the direction of the incoming beam but weakened by the Doppler factor $1 - v_n/c$, while the second represents the tangential drag in question. The total drag in the direction of the beam thus arises from both terms, and is in magnitude $2fv_n/c$, while that in directions transverse to the beam is only fv'/c, where v' is the component of v^α in this direction.*

These approximate equations of motion may be readily interpreted in terms of notions familiar in classical mechanics and electrodynamics. The first force term is, as mentioned above, merely that due to direct radiation pressure on a body whose velocity in the direction of the beam is v_n. Now this particle is absorbing energy at the rate $cf = cAd$, and is re-radiating it in such a way that the outgoing radiation may be thought of as carrying away electromagnetic momentum at the rate $(cf/c^2)v^\alpha$, since it is distributed isotropically about the particle, which is in turn moving with the velocity v^α. But in virtue of the general law of conservation of total (mechanical plus electromagnetic) momentum, this process must cause the particle to lose momentum at the rate fv^α/c, which therefore asserts itself as the retarding force represented by the last term in the equations (3.3).

In the case of a particle moving in the gravitational field of the Sun, absorbing and re-emitting the Sun's radiation falling on it in accordance with the above scheme, we must interpret n^α as the unit vector along the radius vector from the Sun, and take the energy density d as falling off inversely with the square of the distance from the Sun. Let S be the solar constant (radiation falling on a square centimetre normal to the Sun's rays at the mean distance b of the Earth $= 1{\cdot}35 \times 10^6$ ergs per sec. per cm.^{2}); then at the distance r from the Sun

$$d = Sb^2/cr^2, \quad f = mac/r^2, \quad \text{where} \quad a = \frac{ASb^2}{mc^2} = \frac{3Sb^2}{4a\rho c^2}\left(= \frac{2{\cdot}51 \times 10^{11}}{a\rho}\right); \tag{3.4}$$

* Although the additional drag in the direction of the beam is a straightforward consequence of the Doppler factor alluded to above, an effect found by Poynting in a note added to his principal paper on the present question, it seems to have been neglected in most of the treatments referred to in the introductory section. The only paper which has come to my attention in which it has been taken explicitly into account is the second of Plummer's papers on the motion of comets, in which the author thanks Poynting for having called it to his attention. It is conceivable that it is this effect which Larmor had in mind in referring to aberration (p. 756 of Poynting's *C.S.P.*—*cf.* Poynting's somewhat misleading use of the term "aberration," *loc. cit.*, p. 331), but it is to be emphasized that it affects only the component in the direction of the beam, and does not give rise in this approximation to an additional transverse drag.

the second expression for a being that for the case in which we are dealing with a small sphere of radius a, which is nevertheless large compared with the wave-length of the incident radiation, and of material density ρ.* Furthermore, we must of course add on to the right-hand side of (3.3) the force GMm/r^2, in the direction $-n^\alpha$, due to the Sun's gravitational attraction. These equations of motion, in polar co-ordinates (r, θ) in the plane of the orbit with pole at the Sun, are then readily found to be

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} - \frac{2a\dot{r}}{r^2}, \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\frac{a\dot{\theta}}{r}, \quad \text{where} \quad \mu = \mu_0 - ac \tag{3.5}$$

and the dot indicates differentiation with respect to t; μ is the solar gravitational constant $\mu_0 = GM$ reduced by the constant ac representing the repulsive effect of the direct radiation pressure.

These equations of motion are very similar in form to those set up by Poynting and by Plummer, but the numerical coefficients of a, in the two places outside of μ in which this parameter occurs, differ materially from theirs.† However, the formal similarity of the equations will be taken as justification for presenting only a brief account of those aspects of the solution which have also received attention at their hands, emphasizing chiefly the points in which our results differ, and will thus enable us to develop more fully certain other aspects of the problem. Thus it follows, as in Poynting, from the second of the equations (3.5) that the angular momentum H of the particle, per unit mass, decreases in accordance with the law

$$H = r^2\dot{\theta} = h - a\theta, \tag{3.6}$$

where h is its initial value. On employing this integral to eliminate the time in the radial equation, we readily find as the equation of the orbit

$$\frac{d^2u}{d\theta^2} + \frac{a}{H}\frac{du}{d\theta} + u = \frac{\mu}{H^2}, \tag{3.7}$$

where u is the reciprocal $1/r$ of the radius.

Our primary problem is the solution of the equation (3.7) of the orbit, for the problem of determining the position of the particle in the orbit $u = u(\theta)$ at time t is then reduced, in principal, to the solution of (3.6) by a quadrature. Now, although we shall be but little concerned with the rigorous solution of these equations, it is nevertheless of considerable value to show that the orbital equation is reducible to a Bessel equation, for this will enable us to

* We shall in the following express our results in terms of this case of principal interest here; if it is desired to apply them to cases in which this interpretation is inapt, it is only necessary to replace this "effective" $a\rho$ by $3m/4A$.

† For purposes of comparing the results obtained here with those in the literature referred to above, we note that if we characterize the terms $-2a\dot{r}/r^2$, $-a\theta/r$ in the radial and transverse accelerations in (3.5) by the number pair (2, 1), then Poynting's original equations are characterized by the pair (2/3, 2/3), those implied by Poynting in the note appended at the end, and used by Plummer in his second paper, by (4/3, 1/3), those used by Plummer in his first paper by (1/3, 1/3), and those implied by Larmor's work by (1, 1), or by (2, 2) if his aberrational effect is taken into account.

obtain the more readily a general notion of the behaviour of the exact solutions, which will in turn be of considerable assistance in judging the range in which the more manageable approximate solutions, dealt with in the sequel, are valid. This reduction is accomplished by subjecting the variables (u, θ) to the simple transformation

$$\theta = h/a - \phi, \qquad u = \mu\phi z/a^2 \tag{3.8}$$

to new variables (z, ϕ); the equation (3.7) of the orbit then becomes

$$\frac{d^2z}{d\phi^2} + \frac{1}{\phi}\frac{dz}{d\phi} + \left(1 - \frac{1}{\phi^2}\right)z = \frac{1}{\phi^3}, \tag{3.9}$$

an inhomogeneous equation for z whose complementary function is a Bessel function of order 1 and argument ϕ. The equation (3.6), determining the position of the particle at time t, may then be written in the convenient form

$$\dot{\phi} + \alpha u^2\phi = 0. \tag{3.10}$$

In the applications of greatest interest we shall find that $\phi >> 1$, which suggests the use of the well-known asymptotic expansions for the complementary function,* and of a corresponding expansion for the particular integral, to which we now turn our attention. From this standpoint the simplest particular integral—which is at the same time the most interesting for the following—is that one which does not partake of the characteristic oscillatory nature of the Bessel functions for large values of the argument ϕ; the asymptotic series for this unique integral, for values of $\phi >> 1$, is given by the divergent series

$$\phi z \sim \frac{1}{\phi^2}\left\{1 - \frac{8}{\phi^2} + \cdots + \frac{(-)^n 2^{2n} n!\,(n+1)!}{\phi^{2n}} + \cdots\right\}, \tag{3.11}$$

as may be verified by substitution in (3.9). Although of little practical application, it is of some theoretical interest to remark that *any* solution of (3.7) can be shown to assume the form

$$\phi z = P(\phi) + Q(\phi)\log\phi + \tfrac{1}{4}\phi\,(\log\phi)^2 \mathfrak{J}_1(\phi), \tag{3.12}$$

where $\mathfrak{J}_1$ is the Bessel coefficient of order 1 and P, Q are even functions which are analytic in the neighbourhood of the origin $\phi = 0$. The leading terms in the expansion of u, or rather of the function ϕz which only differs from it by a constant factor, are then given by

$$\phi z = -\tfrac{1}{2}\log\phi - 2A + \tfrac{1}{8}\phi\,(\log\phi)^2 + A\phi^2\log\phi + B\phi^2 + \cdots, \tag{3.13}$$

where A and B are arbitrary constants involved in the complementary function; it is to be observed that the first term, which is of most importance for sufficiently small ϕ, is common to *all* solutions of the orbital equation.

One immediate consequence which can be drawn from this last result is

* For the properties here used of the Bessel functions, see E. T. Whittaker and G. N. Watson, *Modern Analysis*, Chap. 17 (5th ed., Cambridge, 1920); the asymptotic series in question are developed in Sections 17.5, 17.6.

that, in case $\mu \neq 0$, r cannot remain indefinitely within finite non-zero limits ; the particle must either fall into the Sun or escape to infinity. For if we assume that r does lie within such limits, it follows from (3.10) that $\phi \to 0$ as $t \to \infty$, and hence from the dominant term in (3.13) that $\phi z \to \infty$, which is contrary to assumption. This means, in the case $\mu > 0$, in which the direct radiation pressure is not enough to reverse the direction of effective gravity, that the particle either escapes completely from the solar system ($\phi z \to 0$) or is drawn into the Sun (ϕz sufficiently large), and in the case $\mu < 0$, in which effective gravity is reversed, that it is blown away, unless its initial circumstances of motion are such that it happens to hit the solar surface first. For completeness we note that the physically unimportant exceptional case $\mu = 0$, in which gravity is just annulled by radiation pressure, allows the further possibility that the particle is in or approaches the unstable limiting position $r = \text{const.}$, $\phi = 0$. In short, there are no finite orbits in the cases $\mu \neq 0$, for all either intersect the Sun's surface or go to infinity—and even in the exceptional case $\mu = 0$, there is in addition only a possible motion of limitation.

In the applications α/h is small, and for many purposes it suffices to retain only terms of first order in this ratio. The equation (3.7) of the orbit then reduces to

$$\frac{d^2u}{d\theta^2} + \frac{\alpha}{h}\frac{du}{d\theta} + u = \frac{\mu}{h^2}\left(1 + 2\frac{\alpha}{h}\theta\right), \tag{3.14}$$

and is readily handled by standard perturbation theory.

4. *The Particular Integral. Resisting Medium.*—The first and most striking consequence of the retardation is that stressed by Poynting—the gradual loss of angular momentum will eventually result in any particle being swept into the Sun, provided only that the solar gravitational pull is not counterbalanced by radiation pressure. It follows immediately from (3.6) that such a particle cannot make as many as $h/2\pi\alpha$ revolutions about the Sun ; if we take as the initial value h that for a circular orbit of radius r, under effective gravity μ, this upper limit is given by

$$N = \frac{(\mu r)^{\frac{1}{2}}}{2\pi\alpha} = 2{\cdot}82 \times 10^7 a\rho(R\mu/\mu_0)^{\frac{1}{2}}, \tag{4.1}$$

where R is the initial distance r (in cm.) expressed in astronomical units. Thus a particle of radius 1 cm. and density 5·5, starting from a circular orbit at the Earth's distance, cannot make as many as $1{\cdot}55 \times 10^8$ revolutions, in substantial agreement with the conclusions reached by Poynting.

In order to obtain an estimate of the times involved, we consider the motion of a particle whose orbit is given by the particular integral (3.11) of the equation of the orbit. Now the magnitude of the second term in the expansion of z is greatest when ϕ is least, and the least value which ϕ can assume is that when the particle is at the surface of the Sun ; this is essentially the value of h/α for a circular orbit at the distance $R = 4{\cdot}65 \times 10^{-3}$ A.U. of the Sun's surface. But here, by (4.1), the ratio $1 : \phi^2$ is only $6{\cdot}9 \times 10^{-15}\mu_0/\mu(a\rho)^2$, and for the extreme case in which the direct radiation pressure is almost

sufficient to counterbalance gravity—for which the effective $a\rho \sim 5{\cdot}7 \times 10^{-5}$—the ratio is still only of the order $2{\cdot}1 \times 10^{-6}\mu_0/\mu$. Hence, unless the pressure is very close indeed to the solar gravitational pull, we may safely take as the equation of the orbit the asymptotic expansion (3.11); carrying only the first two terms of this series, we find as the equation of the spiral orbit

$$r = a^2\mu^{-1}(\phi^2 + 8 + \ . \ . \ .), \tag{4.2}$$

where the next term is of the order ϕ^{-2}. On expressing ϕ in terms of r, the differential equation (3.10) assumes the form

$$2a\,dt + r(1 + 8a^2/\mu r + \ . \ . \ .)\,dr = 0, \tag{4.3}$$

and on integrating this equation we find that a particle initially at the distance r from the Sun will be at the distance r_1 at time

$$t = (r^2 - r_1^2)/4a + 4a(r - r_1)/\mu + \cdots \tag{4.4}$$

Even for cases in which radiation pressure is a respectable fraction of gravitation—say as much as 99 per cent.—the first term in (4.4) gives a remarkably good approximation to the time it takes a particle to fall from a distance r to the solar surface at r_1. If in addition r is large compared with the Sun's radius, we may further simplify this to the form

$$t = r^2/4a \text{ seconds} = 7{\cdot}0 \times 10^6 a\rho R^2 \text{ years}, \tag{4.5}$$

where R is the initial distance r expressed in astronomical units. An illuminating alternative form of this result is obtained by expressing the time of fall t in terms of the period P of the initial, essentially circular, orbit described by the particle at distance r under the effective μ; we may then write

$$t = \tfrac{1}{4}NP, \quad \text{where} \quad P = 2\pi\mu^{-\frac{1}{2}}r^{\frac{3}{2}}, \tag{4.6}$$

and N is the number of revolutions (4.1) in which the particle falls into the Sun.

To give some numerical results, we note that it follows from (4.5) that a particle of radius 1 cm. and density 5·5 (that of the Earth), starting from the distance of the Earth, will be swept into the Sun in $3{\cdot}9 \times 10^7$ years, while a similar particle of radius 10^{-3} cm. can survive but $3{\cdot}9 \times 10^4$ years. The lower limit for the time of fall of a particle from the distance of the Earth is of the order of 400 years, and is realized by a particle whose effective $a\rho$ is near, but not too close to, the critical value $5{\cdot}7 \times 10^{-5}$ at which radiation pressure just annuls the solar gravitation. Alternatively, we may express the result above in terms of the size of the region which is swept clear of the particles initially within it in some given period of time t. Thus particles of a *given* $a\rho$ are swept out of a region of radius $11{\cdot}9\ (a\rho)^{-\frac{1}{2}}$ A.U. in 10^9 years; the upper limit of this expression, as the effective $a\rho$ approaches the critical value, is some 1600 A.U., and represents the dimensions of the greatest region in which all particles of *some* effective $a\rho$ (near the critical value) can be swept into the Sun in this time 10^9 years.

It is clear that this effect cannot be ignored in cosmogonic speculations employing a resisting medium, and that it may be expected to rule out certain forms of such a medium. The application to a given type of medium

depends sharply on the nature of the particles constituting it, and turns in the last analysis upon the evaluation of the direct radiation pressure f, for this determines the rate cf at which radiation is absorbed and re-emitted or scattered, and therefore the rate at which angular momentum is lost. The same remarks apply to questions of the origin, nature and continued existence of the medium responsible for the zodiacal light and the counterglow, questions touched upon by Poynting in several of his stimulating addresses on the astronomical consequences of the pressure of light. On the other hand, this effect can hardly be of appreciable influence on the galactic scale, for the time of fall (4.5)—which, it is to be noted, does not obey Kepler's third law—increases so rapidly with distance that no observable effects on interstellar matter are to be expected. This remark applies in particular to bodies describing orbits around the galaxy as a whole (in spite of the fact that the parameter α may then be some 10^{11} times its value for the Sun), and to the sweeping-out of the bulk of the matter constituting the enormous diffuse nebulæ associated with certain stars.

In all of the above we have been concerned with particles whose effective $a\rho$ was not too close to the critical value, defined implicitly by $\alpha c = \mu_0$, at which the direction of effective gravity is reversed—say, no nearer than 1 per cent. for good measure. Although we have with this surely covered all cases of possible astronomical importance, it may nevertheless be not without interest to complete the analysis by adding a few words concerning the behaviour of a particle whose effective $a\rho$ is so small that the asymptotic expansion breaks down. Retaining only the first two terms in the expansion (3.13)—the only ones which do not vanish as $\phi \to 0$—the orbit is the spiral defined by

$$u = -\tfrac{1}{2}\mu a^{-2} \log \phi + \text{const.}, \tag{4.7}$$

and on integrating (3.10) we find as the time taken by such a particle in falling from a distance r to r_1

$$t = 2a(r - r_1)/\mu. \tag{4.8}$$

Hence in the μ-range here under consideration, a further decrease causes the time of fall to increase—toward the value ∞ for $\mu \to 0$, representing in the limit a particle at rest under the Sun's attraction and an equal and opposite radiation pressure. This particular integral u of course vanishes for effective gravity $\mu = 0$, and cannot be used for $\mu < 0$ in the regions considered above, without the addition of a complementary function, for it is there negative.

5. *The Complete Integral. Encke's Comet.*—We conclude with a brief account of the first-order perturbations caused by radiation drag, as obtained by the usual methods from the approximate equation (3.14) of the orbit.* In much of this Section we limit ourselves essentially to a statement of the results, as their derivation follows very closely that of H. C. Plummer † based on the similar perturbing force obtained by Poynting.

* The more accurate long-time treatment, analogous to the development of the previous Section 4 in that it is based on the leading terms in the asymptotic expansion of the solution of the Bessel equation (3.9), becomes here unwieldy—and of course reduces to the present treatment on retaining only terms of first order in a/h.

† *M.N.*, **65**, 229, 1905 ; **67**, 63, 1906.

To this order, then, the equation of the orbit is readily shown to be given by

$$u = p^{-1}\left\{1 + 2\frac{a}{h}\theta + e\left(1 - \frac{a}{2h}\theta\right)\cos(\theta - \gamma)\right\}, \tag{5.1}$$

where $p = h^2/\mu$. Consider first the more important case in which the osculating orbit is an ellipse. The first order secular perturbations, during one revolution, of the elements of this osculating ellipse are then found to be

$$\frac{\Delta p}{p} = -\frac{4\pi a}{h}, \qquad \frac{\Delta e}{e} = -\frac{5\pi a}{h}, \qquad \Delta\gamma = 0,$$
$$\frac{\Delta n}{n} = -\frac{3}{2}\frac{\Delta a}{a} = \frac{2+3e^2}{1-e^2}\frac{3\pi a}{h}. \tag{5.2}$$

From the first of these equations it follows that the semi-latus rectum of the ellipse diminishes at the same rate as the distance in the spiral orbit discussed in Section 4 above, and from the second that the eccentricity decreases at an even faster specific rate; not only does the orbit of such a particle spiral in toward the Sun, but it at the same time becomes more circular. The principal modification caused by the inclusion of general relativity, for times short compared with those in which the Schwarzschild metric is appreciably altered because of the radiation of mass by the Sun, is the perihelion advance

$$\frac{\Delta\gamma}{2\pi} = \frac{3\mu^2}{c^2h^2}; \tag{5.3}$$

although this effect is quite comparable in magnitude with those here considered, it would seem to be of little cosmogonic significance.

In the papers referred to above, Plummer has examined the possibility of explaining the anomalous acceleration of the mean motion of Encke's comet, which exhibits itself in a decrease of its period, of about $3\frac{1}{2}$ years, by some $2\frac{1}{2}$ hours on each return. He found, in his second note, that the anomaly could be explained by assuming that the comet's head consisted of particles of the order of 2×10^{-3} cm. radius, but that the effective gravitation would then be reduced, because of the radiation pressure, by one part in 125, and this would lead to an untenable decrease (of the order of one part in 600) of the assumed mean distance. The situation is not much improved by the present revision of Poynting's expressions for the drag; on employing Abbot's value for the solar constant and Backlund's elements for the orbit, as quoted by Plummer, the equations (5.2) yield

$$a\rho = 0{\cdot}016, \qquad ac/\mu_0 = 0{\cdot}0036. \tag{5.4}$$

Thus the particles, of assumed density 5·5, can be some half again as large as in Plummer's work (or twice as large, if we reduce his to the size they would have under the value of the solar constant here used), and the decrease in effective gravity is now of the order of only one part in 280. But this is presumably still inadmissibly large—if we throw all the error on to the assumed mean distance, for example, it would lead to a discrepancy in the parameter of the orbit of the order of one part in 10^4.

For cases in which the direct radiation pressure is so enhanced by diffraction or resonance that it reverses the direction of effective gravity, as in the case of the material constituting the tails of comets, the retardation is correspondingly increased. But here there is but little time for the production of observable effects, for, unless other forces are at work, the particle is lost to the solar system in a relatively insignificant time. If we take the effective μ to represent a repulsion N times as strong as gravity, then

$$\alpha = (N+1)\mu_0/c = (N+1)4{\cdot}41 \times 10^{15}, \tag{5.5}$$

and the equation of the orbit is

$$u = N\mu_0 h^{-2}\left\{e\left(1 - \frac{\alpha}{2h}\theta\right)\cos(\theta - \gamma) - \left(1 + 2\frac{\alpha}{h}\theta\right)\right\}. \tag{5.6}$$

Assume that at perihelion the particle has ν times the circular velocity at that distance ; the constant e, which is essentially the eccentricity of the tangent hyperbola described under the repulsive force $N\mu_0/r^2$ from the same initial conditions, is then found to be

$$e = 1 + \nu^2/N. \tag{5.7}$$

For cases in which this differs but little from unity, the order of the effect is illustrated by the fact that it results in a small decrease η of the semi-angle between asymptotes of amount

$$\eta \sim \frac{\nu^2}{3N}\frac{\alpha}{h}. \tag{5.8}$$

In all cases there is a tendency for the particle to lose energy, as well as angular momentum, to the radiation field—where it is interpretable as Doppler shifts. The mechanical aspects of this may perhaps be most clearly brought out by returning to the equations of motion (3.5) and computing the rate of change of the sum

$$E = T + U = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu_0}{r} \tag{5.9}$$

of the kinetic energy T and the gravitational energy U, per unit mass of the particle. The result of this computation is that

$$\frac{dE}{dt} = -\frac{dV}{dt} - \frac{\alpha}{r^2}(2\dot{r}^2 + r^2\dot{\theta}^2), \quad \text{where} \quad V = -\frac{\alpha c}{r}, \tag{5.10}$$

may be thought of an additional potential energy attributable to the direct radiation pressure, and the second, essentially negative term is that due to the non-conservative dragging forces. It therefore follows that the sum $T + U + V$ must decrease monotonically ; but then on describing a closed circuit, or an arc beginning and ending at the same distance r from the Sun, the kinetic energy T must alone suffer the entire loss, since U and V return to their original values. Thus, in particular, a particle entering the solar system from without must leave with a smaller limiting velocity—if, indeed, it ever gets out again, for unless its entering velocity is sufficiently in excess of the parabolic velocity under effective gravity μ, it will be captured by the Sun.

6. *Summary.*—This examination, from the standpoint of the theory of relativity, of the problem of the retardation of motion of a body receiving and re-emitting radiation, has established the existence of a drag of the same general nature as that predicted by Poynting on classical grounds. The actual expressions for the retarding force, for the case of a spherical particle reflecting or absorbing and re-emitting the radiation it receives from the Sun, differ from those obtained by Poynting and by Larmor, even in the approximation to which their considerations were restricted, and it was therefore considered desirable to supplement the rigorous derivation by a brief account of the specific effects arising from the drag in the justifiable approximation in which only terms of first order in the velocity are retained. Those effects of greatest astronomical interest would seem to be the secular decrease of the mean distance of the particle from the Sun—of roughly three times the amount predicted by Poynting and by Plummer, or one-half of that implied by the later considerations of Larmor—and the similar decrease in the eccentricity of the osculating orbit, all on the supposition that the direct radiation pressure is not strong enough to reverse the direction of effective gravity. The point of most significance for cosmogony is that this drag, attendant on the pressure due to the solar radiation, constitutes a force which is effective in clearing the neighbourhood of the Sun of small particles in astronomically significant times.

Princeton University :
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TWO VISUAL BINARY ORBITS.

R. v. d. R. Woolley and L. S. T. Symms.

(Communicated by the Astronomer Royal)

This note describes the calculation of the orbits of two visual binary stars which have many features in common. Both orbits are highly inclined and highly eccentric, and in both cases observations in at least one quadrant are lacking, and the important part of the orbit round periastron is unobserved. On the other hand, both stars have completed at least one revolution, so that some elements, notably the period, can be determined with accuracy. Lastly, both stars are approaching periastron, and it is just possible that a careful examination of the spectrum will show doubled lines, as velocity differences of about 20 km./sec. may be expected. Needless to say, if spectroscopic observations of velocity difference are secured they will not only yield excellent data for improving the orbital elements at their weak points, but will also give values for the parallaxes and other important quantities connected with the stars.

In both cases the new orbits have been computed by applying differential corrections to published orbits. The elements of the orbit may be expressed in the form A, B, F, G, e, T, n, such that the rectangular co-ordinates (x, y) of the companion referred to the primary as origin are given by

$$x = AX + FY,$$
$$y = BX + GY,$$

the quantities X, Y being functions of the mean anomaly and eccentricity, and hence of the quantities e, T, n and the epoch. The functions X and Y are tabulated in *Union Observatory Circular*, 71, 1926. The relations between these elements (the Innes elements) and the elements usually given in older publications (Campbell elements) have been stated by van den Bos.*

When the published orbit of a binary system is not greatly in error, differential corrections to these elements may be obtained by computing co-ordinates x and y for the epochs of the normal places adopted, and obtaining differences Δx and Δy in the sense normal place minus computed place. Each such difference gives one equation of condition for the differential correction of the elements of the orbit as under: †

$$X\Delta A + Y\Delta F + P_x\Delta e + Q_x n\Delta T + R_x\Delta n = \Delta x,$$
$$X\Delta B + Y\Delta G + P_y\Delta e + Q_y n\Delta T + R_y\Delta n = \Delta y,$$

where

$$P_x = A\frac{\partial X}{\partial e} + F\frac{\partial Y}{\partial e}, \qquad Q_x = -A\frac{\partial X}{\partial M} - F\frac{\partial Y}{\partial M},$$
$$P_y = B\frac{\partial X}{\partial e} + G\frac{\partial Y}{\partial e}, \qquad Q_y = -B\frac{\partial X}{\partial M} - G\frac{\partial Y}{\partial M},$$
$$R_x = -(t-T)Q_x, \qquad R_y = -(t-T)Q_y.$$

Using ten normal places one obtains twenty equations of condition from which a least square solution may be made, giving the corrections ΔA, etc. These corrections must then be applied to the original elements and a new ephemeris computed. This may or may not agree so well with the normal places that no further approximation is necessary—obviously, this depends on how good the elements with which one started were. If a second approximation is necessary, it is advisable to adjust the elements before starting the second approximation; for example, if the first approximation has overshot the mark one may with advantage come back one-third of the way before starting the second approximation.

In some cases it may be desirable to use the position angle and separation, ρ and θ, rather than rectangular co-ordinates x and y. The equations of condition may be adapted for this purpose quite simply. We have, in fact,‡

$$\rho\Delta\rho = x\Delta x + y\Delta y,$$
$$\rho^2\Delta\theta = -y\Delta x + x\Delta y,$$

from which an explicit evaluation of $\Delta\theta$ and $\Delta\rho$ in terms of ΔA, etc., may be obtained. Notice that $\Delta\theta$ is here in radians, and that it is necessary to weight the equations for $\Delta\theta$ and $\Delta\rho$ properly if they are to be combined.

If the unit for $\Delta\rho$ is 1″·0 and the unit for $\Delta\theta$ is 1°·0, and if the probable

* *Union Obs. Circ.*, 68, 1926.

† van den Bos, *loc. cit.*; see also Woolley and Symms, *M.N.*, **97**, 168, 1937.

‡ van den Bos, *loc. cit.*

error in ρ is 0″·01 and the probable error for θ is 1°·0, the equations for $\Delta\rho$ will have a weight of 10,000 compared with those for $\Delta\theta$, and the equations for $\Delta\rho$ should be multiplied by 100 before combining with those for $\Delta\theta$.

The velocity difference in km./sec. is obtained very simply from the Innes elements and an assumed parallax. We have, in fact,

$$v \text{ in km./sec.} = L\frac{\partial X}{\partial M} + N\frac{\partial Y}{\partial M},$$

where L and N are readily calculated from the quantities A, B, F, G. All the formulæ concerned have been given by van den Bos, but they may be repeated here for completeness.

Let

$$2p = A^2 + B^2 + F^2 + G^2,$$
$$q = AG - BF.$$

Then

$$a^2 = p + \sqrt{(p+q)(p-q)},$$
$$\cos i = (AG - BF)/a^2.$$

These equations give the semi-axis major a and inclination i.

Again, let

$$C^2 = a^2 - A^2 - B^2,$$
$$H^2 = a^2 - F^2 - G^2,$$

with the check $AF + BG + CH = 0$. (The velocity in seconds of arc per annum is $nC\frac{\partial X}{\partial M} + nH\frac{\partial Y}{\partial M}$.) Then if the parallax in seconds of arc is p, we have

$$L = \frac{4{\cdot}737}{p}nC,$$

$$N = \frac{4{\cdot}737}{p}nH.$$

The convention of signs is given in *Union Observatory Circular*, 91, 1934.

OΣ 38 (γ_2 *Andromedæ*)

ADS 1630. R.A. $2^h\ 00^m{\cdot}8$. Dec. $+42°\ 06'$ (1950·0). Magnitudes 5·0, 6·2.

This star is remarkable in that a fairly good orbit can be computed notwithstanding the fact that with one doubtful exception the pair has only been observed over a range of twenty-five degrees in position angle. The period is about fifty-six years, and during fifty-three of these years the companion remains in the twenty-five degrees of position angle close to apastron, describing the remaining three hundred and thirty-five degrees in three years, during which the separation is very small. During the years 1890 to 1892 inclusive only one positive observation was made, by Burnham, who estimated the separation as 0″·05, and as we shall see later there is good reason to reject this observation.

Recent observations which we have used are as follows :—

Date	Position	Separation	No. of Nights	Observer
	°	″		
1928·16	104·8	0·51	2	Furner
29·67	105·1	0·51	3	Phillips
30·08	108·0	0·59	6	Kuiper
31·80	105·3	0·49	7	Phillips
31·95	102·2	0·67	3	Poulkova
31·98	102·7	0·54	3	Hargreaves
32·87	105·2	0·52	2	Hargreaves
32·94	97·8	0·62	3	Poulkova
33·09	106·0	0·49	4	Baize
33·70	105·5	0·52	6	Phillips
33·77	103·9	0·45	3	v. Biesbroeck
34·12	103·1	0·47	4	Baize
35·14	103·4	0·45	3	Baize
36·13	105·0	0·43	4	Baize
36·66	105·5	0·48	3	Phillips
37·07	104·7	0·48	3	Woolley
37·16	102·9	0·42	2	Baize

The last four observations are unpublished, and we have to thank Dr. Baize and the Rev. T. E. R. Phillips very cordially for their kindness in sending us details of their unpublished observations.

The authors of the present note formed normal places out of these and the earlier observations independently of one another, and adopted a compromise in the few cases where the two normal places differed slightly. The adopted normal places are as follows, shown compared with Hussey's orbit computed in 1901, which is the only orbit available except one by Voronov :—*

Date	Normal Places		Hussey's Orbit		Observed minus Computed	
	°	″	°	″	°	″
1848·0	117	0·41	114·4	0·46	+2·6	−0·05
1870·0	108	·57	106·6	0·54	+1·4	+0·03
1881·0	104	·40	100·8	0·35	+3·2	+0·05
1894·7	121	·16	123·2	0·16	−2·2	0·00
1897·5	118	·28	118·3	0·29	−0·3	−0·01
1908·5	113	·49	111·4	0·55	+1·6	−0·06
1914·0	111	·53	109·7	0·59	+1·3	−0·06
1922·2	108·5	·56	107·5	0·57	+1·0	−0·01
1930·5	105·5	·54	104·9	0·47	+0·6	+0·07
1932·8	105·0	·51	102·9	0·40	+2·1	+0·11
1936·2	104·5	·45	102·0	0·34	+2·5	+0·11

There is also Burnham's observation:

1891·7 313° 0″·05,

* *Tashkent Circular*, 27, 1934. Details of this orbit are not available to us, but, according to van Biesbroeck (*Publ. Yerkes Obs.*, 8 (II), 7, 1936), Voronov's orbit gives 100°·1 and 0″·39 for 1933·77 and so is in considerable error at that epoch (and presumably from that time onwards).

which is obviously of great importance in fixing the orbital elements if it is a good observation.

Places computed from Hussey's elements for $O\Sigma$ 38 were compared with the adopted normal places. The differences were fairly large, and large differential corrections were obtained. These corrections depended very largely on whether Burnham's observation was included in the solution or not. After some trials it was found that the elements

A	+·100	F	−·085	e 0·91
B	−·283	G	−·086	n 6·49

$T = 1890{\cdot}5$

represented the observations (normal places) very well, with the exception of Burnham's observation for 1891·7. The position angle at this epoch is bound up with T in particular, and if T is taken as 1891·86, leaving the other elements unaltered, the computed position angle for 1891·70 is 312°, agreeing with Burnham's observation.

The complete list of observations by Burnham between 1888 and 1894 is as follows :—

Date	Observation
1888·60	Only a slight elongation in 120° with the 36-inch.
1889·51	Position 98°·2. Separation 0″·09 (one night).
1890·53	Elongation doubtful. Separation much less than 0″·1.
·57	Seems to be slightly elongated in 304°·6. Distance decidedly less than 0″·1.
·59	Too uncertain.
·66	Elongation if any too slight to measure.
1891·58	Position 310°·7. Too close for distance.
·78	Position 308°·0. Elongation very small. Distance does not seem to be more than 0″·05.
·81	Position 319°·0. Excessively close ; angle only approximate. Distance less than 0″·05.

Barnard's observations are :

	°	″
1893·82	121·8	0·14
1894·56	121·6	0·15
1895·62	118·6	0·18

From these observations it appears that periastron passage probably occurred between 1890 and 1892. To investigate the value of Burnham's observations we have computed two ephemerides from 1888 to 1894, using the values of the six elements A, B, F, G, e and n quoted above, and using two trial values of T. These are compared with the observations below.

Date	Observer and Result		Computed Positions (1) T=1891·86		(2) T=1890·50	
			°	″	°	″
1888·6	Burnham.	Slight elongation	...	0·20	...	0·13
1889·5	Burnham.	98°, 0″·09. 1*n*	94	0·15	84	0·07
1890·6	Burnham.	Elongation doubtful	...	0·09	...	0·03
1891·7	Burnham.	312°, 0″·05	312	0·02	146	0·05
1894·7	Barnard.	121°, 0″·16	125	0·13	123	0·19

(1) If we regard Burnham's position angle of 312° as correct and set $T = 1891{\cdot}86$, it is difficult to see why Burnham succeeded in resolving the pair in 1891 when the separation was only 0″·02, and failed to resolve it in 1890 and 1888 when the separation was much larger—no less than 0″·20 in the latter case. It is impossible to believe that he resolved the pair in 1891 when the separation was 0″·02, and failed on *four nights* in 1890 when the separation was 0″·09.

(2) Rejecting Burnham's position angle for 1891·7 and putting $T = 1890{\cdot}5$ we get more reasonable agreement with his observations of 1888 and 1889, and it is much more reasonable to suppose that the separation was 0″·03 in 1890, the year in which he failed to resolve it. The suggestion here is that he also failed to resolve it in 1891 when the separation was only 0″·05.

We conclude that Burnham's position angle for 1891·7 is spurious. Notice that the same observer suspected an elongation in 305° in 1890·57. The pair could not conceivably have had this position angle in both 1890·5 and 1891·7.

The elements

A	+·100	F	−·085	e	0·91
B	−·283	G	−·086	n	6·49

$T = 1890{\cdot}5$

were now compared with the normal places, and equations of condition for Δx and Δy formed. As Δx and Δy have equal weight in this particular set, there seemed no advantage to be gained by converting into $\Delta\theta$ and $\Delta\rho$. Since the disposition of the normal places is such that little weight is given to the determination of T, two solutions were made, in one of which T was fixed at 1890·5 and in the other T was treated as a variable. (This can be arranged with very little additional labour in solving the normal equations.) The results do not differ very greatly, because the basic elements fit the normal places very well and require only small corrections. The least squares solutions gave in fact the following results :—

Basis		T Fixed		T Variable	
		Correction	Result	Correction	Result
A	+·100	−·001	+·099 ± ·001	+·003	+·103 ± ·001
B	−·283	+·006	−·277 ± ·002	+·002	−·281 ± ·002
F	−·085	+·017	−·068 ± ·014	+·066	−·019 ± ·064
G	−·086	−·070	−·156 ± ·022	−·150	−·236 ± ·244
e	0·91	+·01	0·92 ± ·02	−·01	0·90 ± ·05
n	6·490	−·060	6·430 ± ·020	−·083	6·407 ± ·023
T	1890·5	...	1890·5	−1^{y}·64	1888·86 ± 4·80

Both sets of elements represent the normal places extremely well, with no residual $\Delta\rho$ greater than 0″·02 and no residual $\Delta\theta$ greater than 1°·7 except 1848·0 and 1894·7, which show residuals of about three degrees against both orbits.

Comparison between Orbits and the Normal Positions

	Normal Places		With $T=1890{\cdot}5$				With $T=1888{\cdot}86$			
	θ	ρ	θ	ρ	$\Delta\theta$	$\Delta\rho$	θ	ρ	$\Delta\theta$	$\Delta\rho$
Date	°	″	°	″	°	″	°	″	°	″
1848·0	117	0·41	114·5	0·42	+2·5	−0·01	114·4	0·41	+2·6	0·00
70·0	108	·57	107·6	·56	+0·4	+ ·01	107·9	·57	+0·1	·00
81·0	104	·40	103·7	·42	+0·3	− ·02	104·3	·42	−0·3	− ·02
94·7	121	·16	124·8	·17	−3·8	− ·01	124·0	·18	−3·0	− ·02
97·5	118	·28	119·4	·27	−1·4	+ ·01	119·0	·27	−1·0	+ ·01
1908·5	113	·49	112·7	·49	+0·3	00	112·6	·49	+0·4	00
14·0	111	·53	110·9	·54	+0·1	− ·01	110·9	·55	+0·1	− ·02
22·2	108·5	·56	108·7	·57	−0·2	− ·01	108·8	·58	−0·3	− ·02
30·5	105·5	·54	106·3	·52	−0·8	+ ·02	106·7	·54	−1·2	00
32·8	105·0	·51	105·6	·50	−0·6	+ ·01	106·2	·51	−1·2	00
36·2	104·5	·45	104·2	·44	+0·3	+ ·01	104·8	·44	−0·3	+ ·01

The comparison between the two orbits and the observations round periastron is :

Date	Burnham	$T=1890{\cdot}5$		$T=1888{\cdot}86$	
		°	″	°	″
1888·6	Slight elongation	95	0·15	64	0·02
1889·5	98° 0″·09	88	·08	261	·04
1890·5	Elongation doubtful	290	·02	186	·04
1891·7	(312° 0″·05)	157	·04	145	·07

The orbit with $T=1890{\cdot}5$ agrees rather better with Burnham's negative observations than does the orbit with $T=1888{\cdot}86$. Not much weight should be attached to the single night's observation in 1889, but as far as it goes it supports $T=1890{\cdot}5$. We adopt, therefore, the following elements :—

$$A +{\cdot}099 \quad F -{\cdot}068 \quad e \ {\cdot}92 \qquad T=1890{\cdot}5$$
$$B -{\cdot}277 \quad G -{\cdot}156 \quad n \ 6{\cdot}43$$

We add for convenience

$$\text{Period} = 56{\cdot}0 \text{ years}$$
$$a = 0''{\cdot}32$$
$$i = \pm\ 109°{\cdot}2.$$

The dynamical parallax from the elements is 0″·018.

There is a considerable change from Hussey's elements, especially in the eccentricity e, as is shown by the following comparisons :—

	P	T	e	a	i
	y			″	°
Hussey	55·0	1892·0	0·82	0·35	± 103·4
Present work	56·0	1890·5	0·92	0·32	± 109·2

Ephemeris for Visual Observations

Date	Position	Separation	Date	Position	Separation
	°	″		°	″
1938·0	103·3	0·40	1945·0	93	0·12
40·0	102·1	·35	45·7	86	0·07
41·0	101·1	·31	46·2	52	0·02
42·0	100·4	·27	46·5	290	0·02
43·0	98·8	·23	46·8	256	0·03
44·0	97·0	·18	47·3	190	0·03

The ephemeris shows very clearly why it is hopeless to attempt to get micrometer measures near periastron. Something may be possible with an interferometer, as the star is bright, and it is to be hoped that interferometer measures will be made in 1946 and 1947.

Spectroscopic Observations of Velocity Difference

The two further elements are $C = \mp \cdot 133$, $H = \pm \cdot 274$. The trigonometrical parallax is 0″·005 and the spectroscopic parallax 0″·035 (Schlesinger's *General Catalogue*). In the ephemeris given below we assume that the dynamical parallax of 0″·018 is correct.

Ephemeris for Spectroscopic Observations

Date	Velocity Difference	Date	Velocity Difference
1938·0	∓ 4 km./sec.	1946·5	± 42 km./sec.
45·0	∓ 2 ,,	46·8	± 32 ,,
45·7	± 1 ,,	47·3	± 19 ,,
46·2	± 13 ,,	48·0	± 12 ,,

Notwithstanding the early type of the star (A0) there appears to be a good chance of separating the spectra in 1946. Since T is uncertain by about one year, the time of maximum velocity difference is also uncertain by the same amount.

β 101. 9 Argus

ADS 6420. R.A. $7^h\ 49^m{\cdot}5$. Dec. $-13°\ 46'$ (1950·0). Magnitudes 5·6, 6·2.

Recent observations are as follows :—

	°	″		
1929·71	324·2	0·29	2*n*	Union Obs.
29·92	329·7	0·27	4*n*	Voute
30·19	333·4	0·19	1*n*	Aitken
30·89	342·2	0·21	3*n*	Voute
32·23	355·7	0·16	4*n*	Union Obs.
32·81	24·8	0·17	4*n*	Voute
33·24	35·8	0·16	4*n*	Union Obs.
34·17	57	0·183	7*n*	Union Obs., Interferometer
34·25	56·1	0·20	4*n*	Union Obs.

Making use of these observations and also those published in ADS and βGC, normal places were made independently by each of us. After a discussion of the slight differences between them we adopted the following normal places :—

	°	″
1878·53	302·4	...
1894·80	283·6	0·48
1898·05	292·2	0·63
1902·97	306·3	0·47
1912·50	78·1	0·26
1919·12	289·3	0·51
1923·01	297·6	0·59
1930·20	332·3	0·24
1932·73	(20·0)	0·16
1934·22	55·7	0·19

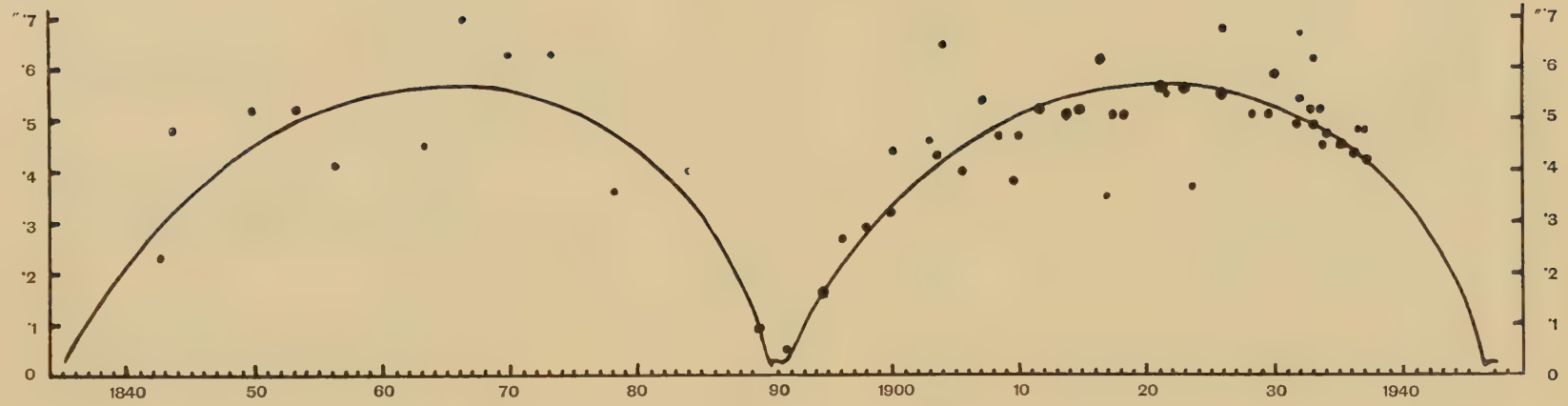

FIG. 1.—OΣ 38. *Observations of separation compared with orbit.*

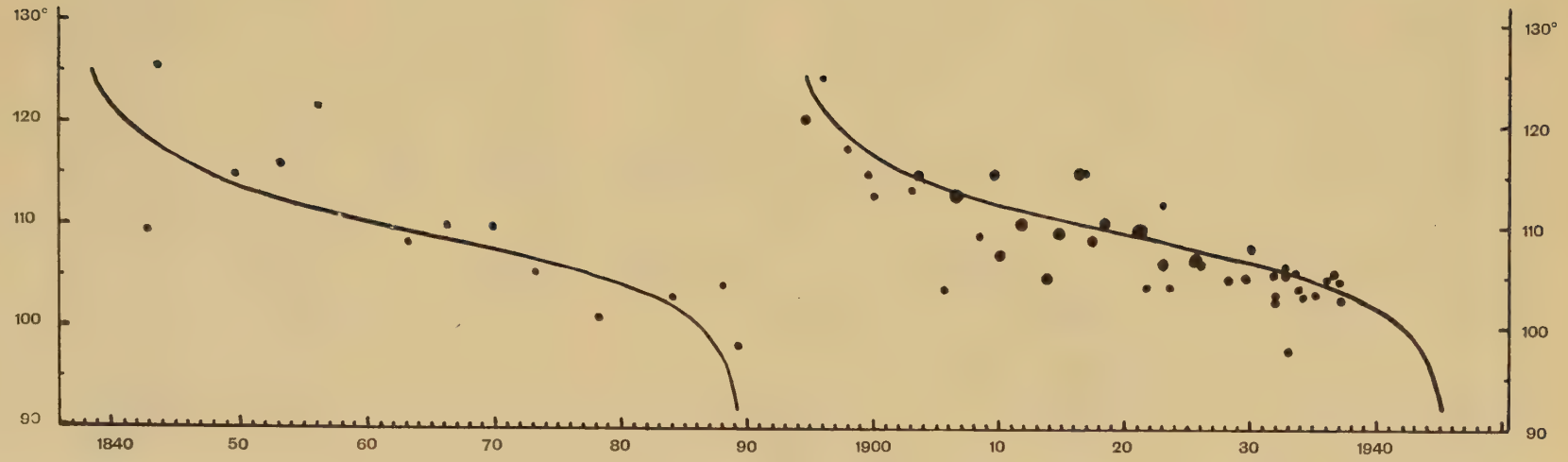

FIG. 2.—OΣ 38. *Observations of position angle compared with orbit.*

Aitken's * orbit was used as a basis for our investigations. This orbit was considered in elliptic rectangular co-ordinates, and differences Δx and Δy

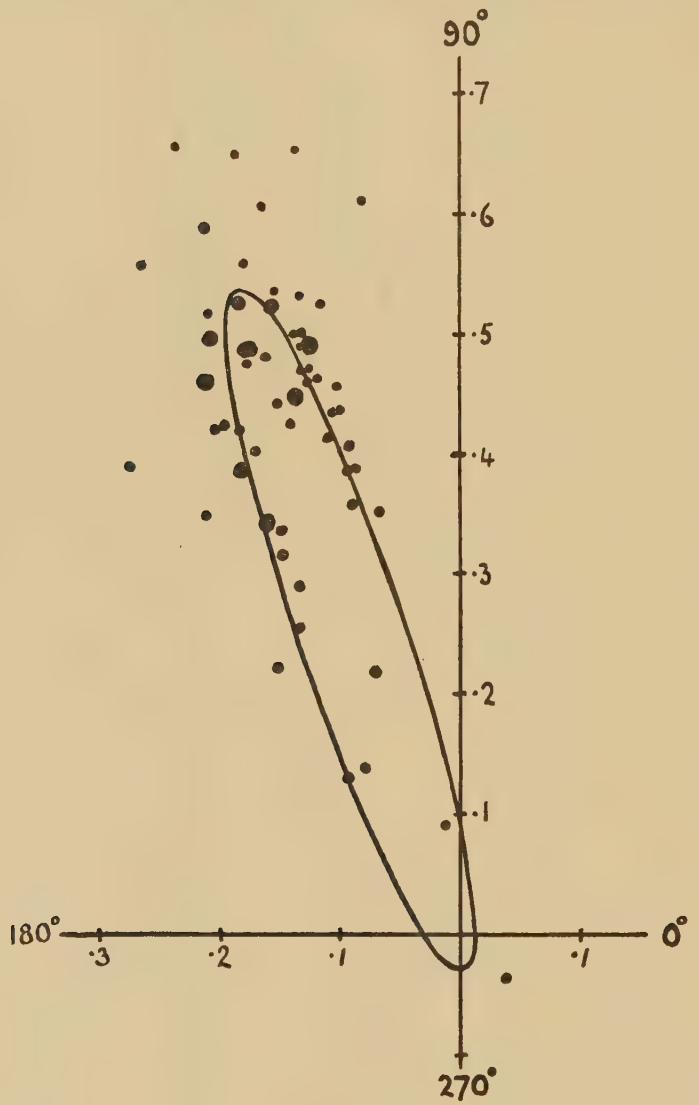

FIG. 3.—*Apparent orbit of OΣ 38.*

were formed between the normal places and the orbit. Corrections to the orbital elements were found by making a least square solution, the results being as follows :—

	A	*B*	*F*	*G*	*e*	*T*	*n*
Aitken's elements	−0·147	+0·160	+0·080	−0·661	0·75	1915·94	15·424
Corrections	−0·010	+0·025	−0·001	+0·173	−0·08	−0·29	− 0·112
Corrected elements	−0·157	+0·185	+0·079	−0·488	0·67	1915·65	15·312

* *Lick Observatory Publications*, **12**, 51, 1914.

On constructing an ephemeris from these corrected elements it was seen that the differential corrections had overshot their mark. In such cases a second approximation is necessary: but before applying the second approximation it is advisable to search for a compromise between the original and first approximation, such as is obtained by going one-third of the way back. It was found by inspection and by constructing a few ephemerides (an easy operation with the elements in this form) that the following elements gave an extremely close representation of the orbit:—

A	B	F	G	e	T	n
−0·156	+0·178	+0·079	−0·546	0·70	1915·80	15·501

They were used to secure a final set of elements by making a least square solution of the residuals ΔA, ΔB, ΔF, ΔG, Δe, ΔT and Δn to fit the residuals $\Delta\theta$ and $\Delta\rho$ (not in this case Δx and Δy) from the normal places (the angle for 1932·73 was omitted). The unit of ρ was taken as 1″·0 and the unit of θ was 1°·0. An examination of the individual observations showed that the probable errors of the normal places are about ± 0″·01 in ρ and ± 1°·0 in θ, so that the equations of condition for $\Delta\rho$ in seconds of arc have a weight 100^2 compared with those for $\Delta\theta$ in degrees. The equations of condition for $\Delta\rho$ were accordingly multiplied by the square root of their weight, or 100, in forming the final normal equations in which both sets of residuals were taken together. (It may be remarked that in this case where the distribution of the observations is peculiar, none being obtained anywhere near periastron, it is quite hopeless to try to form differential corrections from $\Delta\rho$ or $\Delta\theta$ alone, large (and incorrect) corrections of extremely low weight being produced by the process.)

The corrections given by the least square solution were small. They are given, together with their probable errors, as follows :—

ΔA	ΔB	ΔF	ΔG	Δe	ΔT y	Δn
−·004	+·008	−·002	+·018	−·01	−0·09	+·03
±·003	±·015	±·025	±·071	±·05	±1·2	±·04

The concluded elements are accordingly

A −0·160 ± ·003
B +0·186 ± ·015
F +0·077 ± ·025
G −0·528 ± ·071

e 0·69 ± ·05
n 15·53 ± ·04
T 1915·71 ± 1·2

We may add for convenience

Period = 23·18 years
a = 0″·58
i = ± 77°·8

The dynamical parallax from this orbit is 0″·056.

It is remarkable, when one examines a simple plot of the observations in polar co-ordinates, that an orbit can be calculated at all; but in fact Aitken, Löhse, Burnham and Glasenapp (the last mentioned at his second

attempt) have all computed orbits for β101, none of which are substantially in error. Some of the orbital elements are given below.

	P	T	e	a	i
	y			″	°
Aitken *	23·34	1892·60	0·75	0·69	± 79·8
Löhse †	23·48	1892·367	0·7575	0·7100	78·65
Burnham ‡	23·3	1892·7	0·68	0·61	76·6
Glasenapp §	23·33	1892·3	0·668	0·60	77·7
Present work	23·18	1892·53	0·69	0·58	77·8

Burnham's orbit is the closest to ours in most respects. We have, of course, had the advantages afforded by an extra revolution and by some modern observations of separation under 0″·2 which are confirmed by the interferometer.

Comparison of Normal Places with Ephemeris

The concluded orbit gives the following residuals from our adopted normal places, in the sense adopted minus computed :—

Date	$\Delta\theta$	$\Delta\rho$	Date	$\Delta\theta$	$\Delta\rho$
	°	″		°	″
1878·53	+0·5	...	1919·12	+0·9	−0·03
1894·80	−1·2	+0·02	1923·01	+0·6	+0·01
1898·05	−0·9	+0·03	1930·20	+0·1	−0·02
1902·97	+0·6	0·00	1932·73	(−1·6)	−0·01
1912·50	−0·4	0·00	1934·22	−0·8	−0·01

Ephemeris for Visual Observations

Date	Position	Separation	Date	Position	Separation
	°	″		°	″
1935·03	69·9	0·23	1939·21	241·5	0·06
36·32	85·0	0·28	39·53	269·4	0·15
37·60	96·9	0·29	40·18	279·6	0·31
38·25	104·3	0·23	41·47	286·0	0·49
38·57	110·9	0·16	42·75	289·6	0·57
38·89	130·7	0·08	44·68	293·9	0·60

Spectroscopic Observations of Velocity Difference

The two further elements are $C = \pm\ \cdot520$, $H = \pm\ \cdot214$. The trigonometrical parallax is 0″·060 (Schlesinger's *General Catalogue*), and the spectroscopic parallax 0″·063 (Mount Wilson). If we assume that the dynamical parallax of 0″·056 is correct, we find $L = 681$, $N = 280$, and obtain the following ephemeris for the velocity difference :—

Ephemeris for Spectroscopic Observations

Date	Velocity Difference	Date	Velocity Difference
1937·60	± 20 km./sec.	1939·53	∓ 7 km./sec.
38·25	± 22 ,,	40·18	∓ 12 ,,
38·57	± 20 ,,	41·47	∓ 13 ,,
38·89	± 12 ,,	42·75	∓ 11 ,,

The spectrum is of type Go.

* *Publ. Lick Obs.*, **12**, 51, 1914.
† *Publ. Potsdam Obs.*, **20**, 93, 1908.
‡ *Publ Lick Obs.*, **2**, 239, 1894.
§ *M.N.*, 54, 318, 1894.

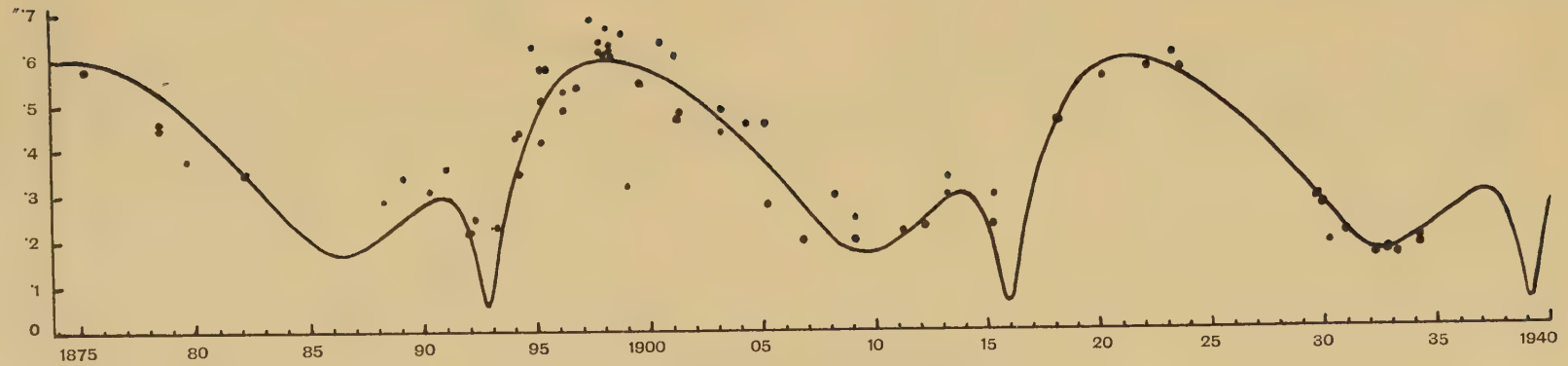

FIG. 4.—β 101. *Observations of separation compared with orbit.*

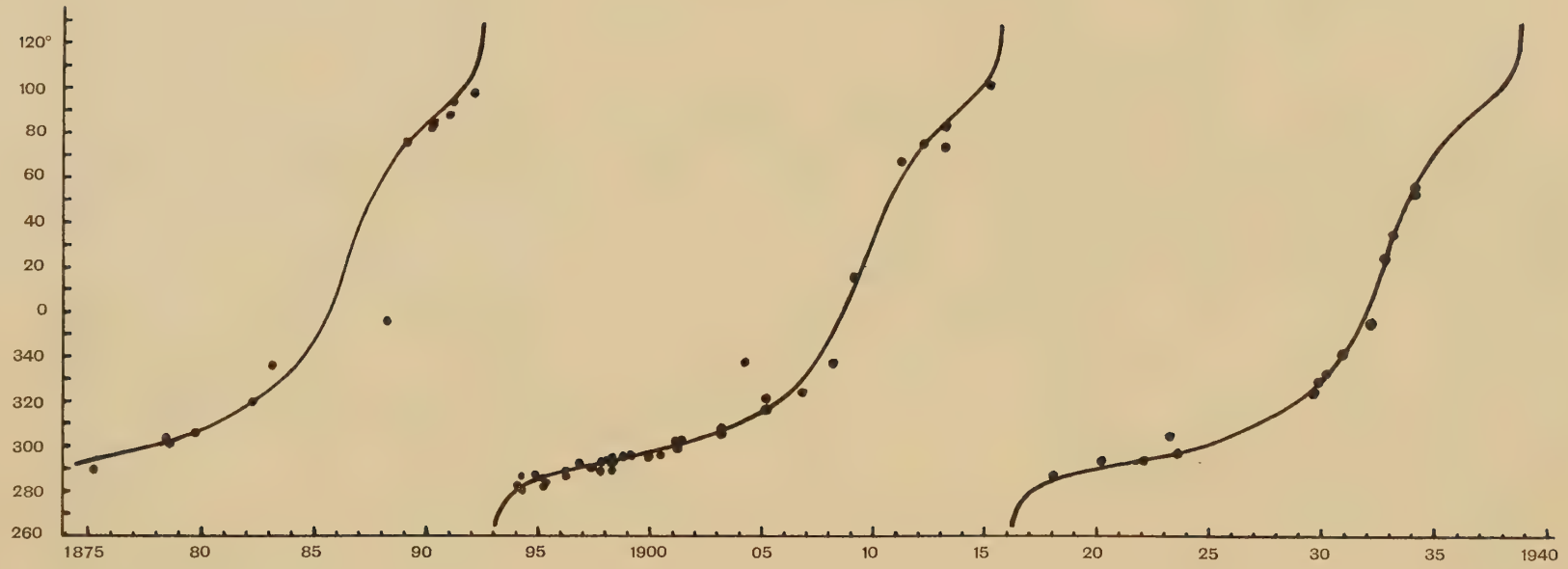

FIG. 5.—β 101. *Observations of position angle compared with orbit.*

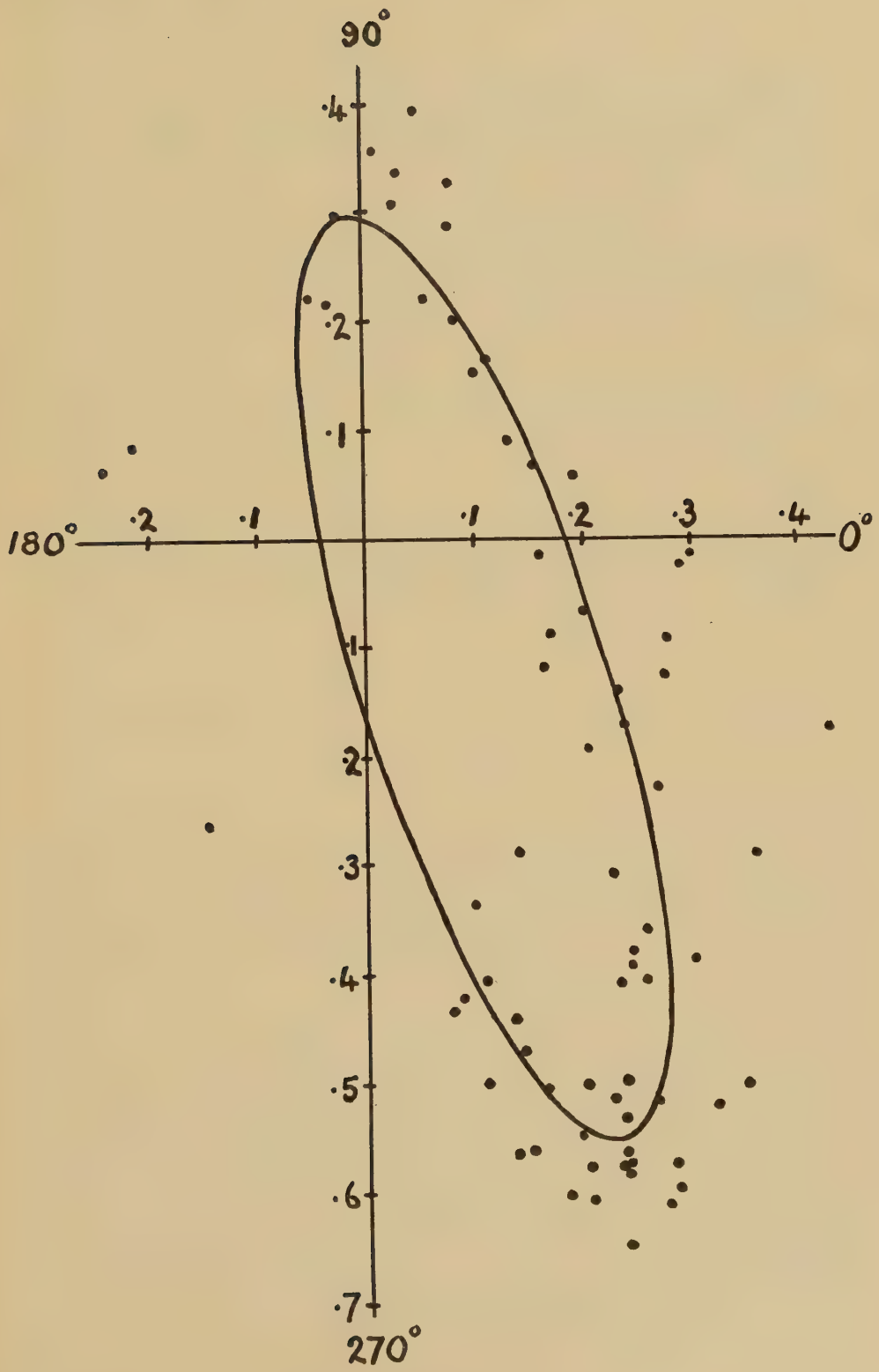

FIG. 6.—*Apparent orbit of* β 101.

ON THE STATISTICAL DETERMINATION OF THE MASSES AND PARALLAXES OF BINARY STARS.

C. Barnes, Ph.D.

(Communicated by Professor C. A. Chant)

If for a double star we know the semi-axis major a of the true orbit (in seconds of arc) and the period P (in years), the combined mass m_1+m_2 of the system (with the Sun's mass as unit) is given by the familiar expression

$$m_1+m_2=a^3/P^2p^3, \tag{1}$$

where p is the parallax (in seconds of arc). Conversely, if the combined mass is known or can be estimated, the same relation may be used to find the parallax. In most double-star observations, however, the data are insufficient to determine a and P, and we are restricted to measurements of a limited arc in the apparent ellipse. From these measurements the mass and parallax may be estimated by the statistical processes of Russell and of Hertzsprung. If we denote the apparent separation of the components by ρ (seconds of arc) and the apparent relative motion by w (seconds of arc per year), the formulæ used are *

$$m_1+m_2=\rho w^2/17p^3 \tag{2}$$

for an estimate of the combined mass with a probable error of about 49 per cent. and

$$p=0{\cdot}42\left(\frac{\rho w^2}{m_1+m_2}\right)^{1/3} \tag{3}$$

for an estimate of the parallax with a probable error of about 20 per cent. The numerical values of the constants in these formulæ can be found from stars of known parallax (Hertzsprung, Schwarzschild), or they can be obtained theoretically by calculating the mean values of ρw^2 and $(\rho w^2)^{1/3}$. Russell does not appear to have published the calculations on which he intimates (2) and (3) to be based, and a discussion of the problem by Jackson and Furner † is (they admit) not rigorous.

In averaging ρw^2 and $(\rho w^2)^{1/3}$ it is fairly reasonable to assume that for an orbit of given eccentricity all directions of the normal to the plane of the orbit are equally probable, and that in the plane of the orbit all directions of the major axis are equally probable. All possible values of the eccentricity are certainly *not* equally probable. In this paper will be submitted the mean values of ρw^2 and $(\rho w^2)^{1/3}$ for orbits of given eccentricity; also the mean values of $(\rho w^2)^2$ and $(\rho w^2)^{2/3}$ required for the estimation of the probable errors. The mean eccentricity required for the final evaluation of the constants in (2) and (3) can then be introduced, its value being derived from observational data.

* *Cf.* Russell, Dugan, Stewart, *Astronomy*, **2**, 693 (1927).

† *M.N.*, **81**, 2.

1. *Mean Values of* ρw^2 *and* $(\rho w^2)^2$

Take a set of rectangular axes x, y, z with the origin in m_1, one of the two gravitating masses m_1, m_2, and with the z-axis in the line of sight, positive away from the observer. The relative ellipse described by m_2 lies in a plane making an angle i with the xy-plane. Let the x-axis be taken through the ascending node of the plane of the orbit, and the positive direction of the y-axis such that a rotation from the x-axis to the y-axis is in the direction of motion of m_2. Let ω be the angle from the node to the periastron, r the radius vector from the origin to m_2, and V the (relative) velocity of m_2. Then we have

$$\rho w^2 = rV^2 \cos\phi \cos^2\phi', \tag{4}$$

where ϕ, ϕ' are the angles between the xy-plane and the vectors r, V respectively. If α is the angle between the vectors r, V, and u is the angle between the plane of the orbit and the plane through r perpendicular to the xy-plane, we have

$$\cos^2\phi' = 1 - (\cos\alpha \sin\phi - \sin\alpha \cos\phi \cos u)^2. \tag{5}$$

The distribution in i, ω (of the kind postulated above) can be replaced by one in ϕ, u. By direct change of variables the elementary probability $\sin i di d\omega$ becomes $\cos\phi d\phi du$, and the mean value of $(\rho w^2)^n$ for orbits with the same α, r, V is

$$(rV^2)^n \frac{1}{\pi}\int_0^{\pi/2}\int_0^{\pi} \cos^{n+1}\phi \cos^{2n}\phi' d\phi du, \tag{6}$$

with the value of $\cos^2\phi'$ form (5). For $n=1$ this gives

$$rV^2 \cdot \frac{3\pi}{16} \cdot (1 - \tfrac{1}{6}\sin^2\alpha); \tag{7}$$

and for $n=2$ it gives

$$(rV^2)^2 \cdot \tfrac{16}{35} \cdot (1 - \tfrac{1}{3}\sin^2\alpha). \tag{8}$$

The time averages of (7) and (8) must now be found for orbits of given eccentricity e. If T, R denote the transverse and radial components of V,

$$\sin\alpha = T/V, \qquad \cos\alpha = R/V.$$

We have

$$rV^2 \sin^2\alpha = rT^2 = \frac{4}{r}\left(\frac{\pi a^2}{P}\right)^2 (1 - e^2),$$

if a, P are respectively the semi-axis major and period of the motion. The mean of $1/r$ is $1/a$, and that of rT^2 is therefore $4\pi^2 a^3(1-e^2)/P^2$. The velocity of the motion is given by

$$V^2 = \frac{4\pi^2 a^3}{P^2}\left(\frac{2}{r} - \frac{1}{a}\right),$$

and using the value $a(1+e^2/2)$ as the mean of r, the mean of rV^2 is $4\pi^2 a^3(1-e^2/2)/P^2$. Finally, then,

$$\overline{(\rho w^2)} = \tfrac{5}{8}\pi^3(1 - \tfrac{2}{5}e^2)\frac{a^3}{P^2}. \tag{9}$$

The time averages of $(rV^2)^2$ and $(rV^2)^2 \sin^2 \alpha$ are similarly found to be respectively

$$\left(\frac{4\pi^2 a^3}{P^2}\right)^2 (1 - \tfrac{1}{2}e^2), \qquad \left(\frac{4\pi^2 a^3}{P^2}\right)^2 (1 - e^2),$$

and hence from (8)

$$\overline{(\rho w^2)^2} = \tfrac{32}{105}\left(\frac{a^3}{P^2}\right)^2 16\pi^4 (1 - \tfrac{1}{4}e^2). \tag{10}$$

2. *Mean Value of* $(\rho w^2)^{1/3}$

For $n = 1/3$ we must evaluate the integral

$$\int_0^{\pi/2}\int_0^{\pi} \cos^{4/3}\phi [1 - (\cos\alpha \sin\phi - \sin\alpha \cos\phi \cos u)^2]^{1/3} d\phi du. \tag{11}$$

This integral, regarded as a function of α, has a minimum value for $\alpha = 90°$ and a maximum value for $\alpha = 0°$. The maximum value is

$$\pi \int_0^{\pi/2} \cos^2\phi d\phi = 2{\cdot}467,$$

and the minimum value is

$$2\int_0^{\pi/2}\int_0^{\pi/2} \cos^{4/3}\phi (1 - \cos^2\phi \cos^2 u)^{1/3} d\phi du = 2{\cdot}382 \tag{12}$$

(by numerical integration), a total variation of about $3\frac{1}{2}$ per cent. We will retain (12) as the value of (11) and show in the next section that the error so introduced is quite negligible.

The time average of $(rV^2)^{1/3}$ is, in terms of the eccentric anomaly E,

$$\left(4\pi^2 \frac{a^3}{P^2}\right)^{1/3} \frac{1}{\pi}\int_0^{\pi} (1 + e\cos E)^{1/3}(1 - e\cos E) dE = \left(4\pi^2 \frac{a^3}{P^2}\right)^{1/3} (1 - \tfrac{2}{9}e^2 - \tfrac{25}{648}e^4),$$

higher powers of e being negligible. Finally,

$$\overline{(\rho w^2)^{1/3}} = 2{\cdot}382 \left(\frac{a^3}{P^2}\right)^{1/3} \left(\frac{4}{\pi}\right)^{1/3} (1 - \tfrac{2}{9}e^2 - \tfrac{25}{648}e^4). \tag{13}$$

3. *Numerical Values and Probable Errors*

The distribution of values of the eccentricity could be determined only if the mode of formation of a binary star were clearly recognized. Lacking this information, however, it is impossible to regard all values of e between 0 and 1 as equally probable, and we are compelled to take the mean value of e provided by measured orbits. With the values *

$$\overline{e^2} = 0{\cdot}322, \qquad \overline{e^4} = 0{\cdot}14$$

we get

$$\overline{\rho w^2} = 16{\cdot}9(a^3/P^2), \tag{14}$$

$$\overline{(\rho w^2)^{1/3}} = 2{\cdot}38(a^3/P^2)^{1/3}. \tag{15}$$

* From 124 orbits, as computed by Professor R. K. Young.

We may recall at this stage that the numerical coefficient in (15) is derived from the minimum value (12) of the integral (11). Now for orbits with $e < 0{\cdot}6$ it is easily shown that during the motion the angle α varies between $53°$ and $90°$, and a simple calculation then shows that the error introduced by the use of (12) is less than 1 per cent. Such an error is quite without significance in (15), and no correction need be applied.*

From (10)

$$\overline{(\rho w^2)^2} = 437(a^3/P^2)^2. \qquad (16)$$

The dispersion of the values of $\rho w^2/(a^3/P^2)$ is then

$$\sqrt{[437 - (16{\cdot}9)^2]} = 12{\cdot}3,$$

and the probable error in a single determination is

$$12{\cdot}3 \times 0{\cdot}6745/17 = 49 \text{ per cent.} \qquad (17)$$

We can find $(\rho w^2)^{2/3}$ with all the accuracy needed here by interpolation in the series of logarithms of 437, 16·9, 2·40, 1·0, the coefficients corresponding to the powers 2, 1, 1/3, 0 :

$$\overline{(\rho w^2)^{2/3}} = 6{\cdot}2(a^3/P^2)^{2/3}.$$

The dispersion of the values of $(\rho w^2)^{1/3}/(a^3/P^2)^{1/3}$ is

$$\sqrt{[6{\cdot}2 - (2{\cdot}38)^2]} = 0{\cdot}735,$$

and the probable error in a single determination is

$$0{\cdot}735 \times 0{\cdot}6745/2{\cdot}4 = 21 \text{ per cent.} \qquad (18)$$

We see that the results (14), (15), (17), (18) are in complete agreement with the formulæ quoted in the introduction. Even if a somewhat different mean value of e be taken, the final results are essentially unchanged. For instance, with the values †

$$\overline{e^2} = 0{\cdot}263, \qquad \overline{e^4} = 0{\cdot}11,$$

the numerical coefficient in (14) becomes 17·3, and that in (15) becomes 2·42.

I must here express my indebtedness to Professor R. K. Young for his invaluable advice and criticism.

University of Toronto :
1937 *January* 2.

* The value of (11) for $\alpha = 55°$ is 2·407.

† Derived from the table on p. 291 of Jeans, *Astronomy and Cosmogony*, Cambridge, 1929.

THE NON-UNIFORM ROTATION OF THE SUN AND ITS MAGNETIC FIELD.

V. C. A. Ferraro, Ph.D.

1. *Introduction and Summary.*—The problem of the uniform rotation of the Sun in its own magnetic field has been considered by Cowling ; * the case considered by him was essentially that of a mass of ionized gas rotating as a rigid body in the presence of an axially symmetrical magnetic field whose axis coincided with the axis of rotation. Cowling showed that in this case a polarisation of the gas would be set up, but it is to be noted that this could only subsist if the magnetic field vanishes over the surface of the gas ; for the surface charges that would otherwise be induced would quickly and continually escape into the surrounding space under the electrostatic repulsion of the charges of like sign. A flow of electric currents in meridianal planes would thereby be set up, tending to maintain the surface-charge density.

The purpose of this paper is to consider the more general problem of the *non-uniform* rotation of the Sun in its own magnetic field, and particularly its relation to the problem of the radial limitation of this field. It will be shown that in the general case, in addition to a polarisation, there is in general a tendency for electric currents to flow in meridianal planes unless the angular velocity is constant over any surface traced out by a complete rotation of a magnetic line of force around the axis of rotation.†

It will be shown that for the Sun the latter alternative must hold to a high degree of approximation, and the bearing of this result on the problem of the radial limitation of the magnetic field is discussed : it is suggested that the diminution of this field radially outwards is unlikely to be as rapid as has been generally supposed.

2. *An Illustrative "Slab" Problem.*—We begin by considering the following two-dimensional problem which will serve to illustrate the main features of the rotational problem : consider a neutral ionized gas in the form of an infinite slab having its plane faces parallel to, and moving in, a magnetic field **H**, uniform in direction, whose intensity is constant over planes parallel to the faces of the slab. The velocity, **v**, of the particles in the stream is perpendicular to **H** at every point. The magnetic field tends to deflect the ions and electrons in opposite directions and separate them, and this tendency is opposed by the electrostatic field set up by the separation of the charges of unlike signs. Let us consider firstly an analogue of Cowling's problem in which a polarisation of the slabs is all that results. In this case **v** must, in addition, be constant over planes parallel to the

* *M.N.*, **90**, 140, 1929.

† The slight inclination of the magnetic axis to the axis of rotation of the Sun will not be considered in this paper.

faces of the slab ; thus we have

$$\mathbf{v} = \mathbf{f}(\mathbf{H}), \tag{1}$$

f being any vector function.

Choose a system of Cartesian axes, with the z-axis in the direction of **H**, the x-axis parallel to the faces of the slab, the origin being at a point midway between the faces ; then the only component of v is the x-component, and so

$$\mathbf{v} = (v, 0, 0). \tag{2}$$

Let **E** denote the electrostatic field * resulting from the polarisation of the stream, and let m and e denote respectively the mass and electric charge of a particle in the stream. The equation of motion of this charge is then

$$m\dot{\mathbf{v}} = e\mathbf{E} + e\mathbf{v}_\wedge\mathbf{H}. \tag{3}$$

In the steady state in which only a polarisation of the stream results we must have

$$\mathbf{E} = -\mathbf{v}_\wedge\mathbf{H} ; \tag{4}$$

the motion of the particle is then made up of its mean velocity **v** on which is superposed the velocity $\mathbf{v}_0$ given by

$$\dot{\mathbf{v}}_0 = (e/m)\mathbf{v}_{0\wedge}\mathbf{H} ;$$

the resulting motion of the particle is therefore trochoidal.

The polarisation, **P**, at any point of the stream is given by

$$\mathbf{P} = \frac{1}{4\pi}\mathbf{v}_\wedge\mathbf{H}, \tag{5}$$

whence the volume-charge density is given by

$$-\operatorname{div}\mathbf{P} = -\frac{1}{4\pi}\operatorname{div}(\mathbf{v}_\wedge\mathbf{H}) = \frac{1}{4\pi}\frac{dvH}{dy}, \tag{6}$$

and the surface-charge density over the surfaces of the slab by

$$\begin{aligned} P_n &= v_1H_1/4\pi, \qquad (y > 0) \\ &= -v_2H_2/4\pi, \qquad (y < 0) ; \end{aligned} \tag{7}$$

the suffixes 1, 2 referring to the values of **v** and **H** over the surfaces of the slab. We remark that if v or H or both vanish over these surfaces, P_n also vanishes as has already been mentioned.†

The individual motion of the charges in the surface layers has been considered in previous work to which reference is made.‡

Suppose now that v is not constant over any plane parallel to the surfaces of the slab, but varies, say, with z ; to simplify the argument we shall suppose

* Electromagnetic units are used throughout.

† It is supposed that the density of the stream is sufficiently high to enable the requisite polarisation to be set up.

‡ *Terr. Mag.*, **36**, 83–88, 1931.

also that **H** is constant throughout. If we suppose for the moment that only a polarisation results, we require that (4) should be satisfied. But if we attempt to find the distribution of charges producing the requisite field **E** in this case we meet with a contradiction ; for the space charge, given by (6), is zero, and any distribution of surface charges other than one of uniform density could not produce the field (4), which is uniform in direction. The tendency which the charges of opposite signs have to separate cannot be balanced by the polarisation field, and therefore electric currents will flow in planes perpendicular to the direction of motion.

3. It is of interest to determine the current system and the electrostatic field for this case : it will be assumed that a steady state has been attained.

The electromagnetic equation for the steady field required are, in addition to div $\mathbf{H} = 0$ (which is satisfied by the condition that H is constant over planes $y = \text{const.}$),

$$\mathbf{E} = -\operatorname{grad} V, \tag{8}$$

$$\operatorname{curl} \mathbf{H} = 4\pi \mathbf{J}, \tag{9}$$

$$\mathbf{J} = \sigma(\mathbf{v} \wedge \mathbf{H} - \operatorname{grad} V) + \mathbf{i} ; \tag{10}$$

here V is the electrostatic potential, **J** the total current density at any point and **i** the density of the current system producing **H**, the imposed part of the magnetic field ; σ is the electrical conductivity of the stream supposed constant. Electromagnetic units are used throughout.

The electric currents flowing in planes perpendicular to the direction of motion give rise to a field H' perpendicular to the planes of current-flow ; hence the magnetic field has two components given by

$$\mathbf{H} = (H', 0, H) ; \tag{11}$$

i itself has only an x-component, i say, so that equations (9) to (11) give the following relevant equations :

$$\frac{\partial H}{\partial y} = 4\pi i, \tag{12}$$

$$\frac{\partial H'}{\partial z} = 4\pi \mathfrak{J}_y = 4\pi\sigma(-vH - \partial V/\partial y), \tag{13}$$

$$-\frac{\partial H'}{\partial y} = 4\pi \mathfrak{J}_z = -4\pi\sigma(\partial V/\partial z). \tag{14}$$

Eliminating V from (13) and (14) we have

$$\nabla_1^2 H' = -4\pi\sigma(H \partial v/\partial z). \tag{15}$$

An exact solution of the problem can be obtained in the case when H vanishes over the surfaces of the slab ; for in this case the displacement of the charges over these faces must also vanish, as there is no tendency for the magnetic field to move the charges there. The required solution must be continuous, together with its derivatives, and must vanish, together with $\partial H'/\partial z$, over the faces of the slab ; this last condition ensures that there will be no accumulation of charges there.

To simplify the analysis we shall suppose that v and H are given by

$$H = H_0 \cos py, \tag{16}$$

and

$$v = v_0 \exp(-q\,|\,z\,|), \qquad q \geqslant 0; \tag{17}$$

if the thickness of the slab is taken to be $2a$, the condition that H should vanish over the faces of the slab, $y = \pm a$, is that

$$pa = \pm\pi/2.$$

The assumed law of variation of the velocity v is such that it decreases to zero exponentially at infinite distance from the plane $z = 0$.

The solution of (15) is found to be

$$H' = H_0'\left(\frac{q\delta}{q^2 - p^2}\right)\{\exp(-q\,|\,z\,|) - \exp(-p\,|\,z\,|)\}\cos py; \quad p \neq q \tag{18}$$

$$= -\tfrac{1}{2}H_0' z \exp(-p\,|\,z\,|)\cos py, \qquad p = q \neq 0, \tag{18a}$$

where

$$H_0' = 4\pi\sigma v_0 H_0, \tag{19}$$

$$\delta = 1 \qquad (z > 0)$$
$$= -1 \qquad (z < 0). \tag{20}$$

This solution vanishes over the faces $y = \pm a$, over the plane of symmetry $z = 0$, and for infinite values of $|\,z\,|$; it changes sign with z so that the total magnetic field (that is the imposed field H together with H') is such that the magnetic lines of force are bent forward in the direction of motion of the stream, the curvature of the lines being greatest over the plane $z = 0$ (see fig. 1).

From (13) and (14) it follows that H' is the current-function for the flow of electric currents in the planes perpendicular to the direction of motion of the stream; the current-lines are the family of curves, $H' = \text{const.}$, where H' is given by (18) and (18a); they are symmetrical about the y- and z-axes and intersect the z-axis at right angles, going off to infinity in both the directions parallel to the z-axis (fig. 2).

The components of the current-density of the cross-currents are found to be

$$\mathcal{J}_y = \sigma v_0 H_0\left(\frac{q}{q^2 - p^2}\right)\{p\exp(-p\,|\,z\,|) - q\exp(-q\,|\,z\,|)\}\cos py, \tag{21a}$$

$$\mathcal{J}_z = \sigma v_0 H_0\left(\frac{pq\delta}{q^2 - p^2}\right)\{\exp(-q\,|\,z\,|) - \exp(-p\,|\,z\,|)\}\sin py, \tag{21b}$$

and it can be shown that $\operatorname{div}\mathbf{J} = 0$, as should be the case.

The components of the electrostatic field are

$$E_y = -\frac{\partial V}{\partial y} = v_0 H_0\left(\frac{p}{q^2 - p^2}\right)\{q\exp(-p\,|\,z\,|) - p\exp(-q\,|\,z\,|)\}\cos py, \tag{22a}$$

$$E_z = -\frac{\partial V}{\partial z} = v_0 H_0\left(\frac{pq\delta}{q^2 - p^2}\right)\{\exp(-q\,|\,z\,|) - \exp(-p\,|\,z\,|)\}\sin py. \tag{22b}$$

It is seen from (21*a*) that $\mathcal{J}_y$ vanishes over the faces of the slab, so that there is no accumulation of charge over these : also there are no surface

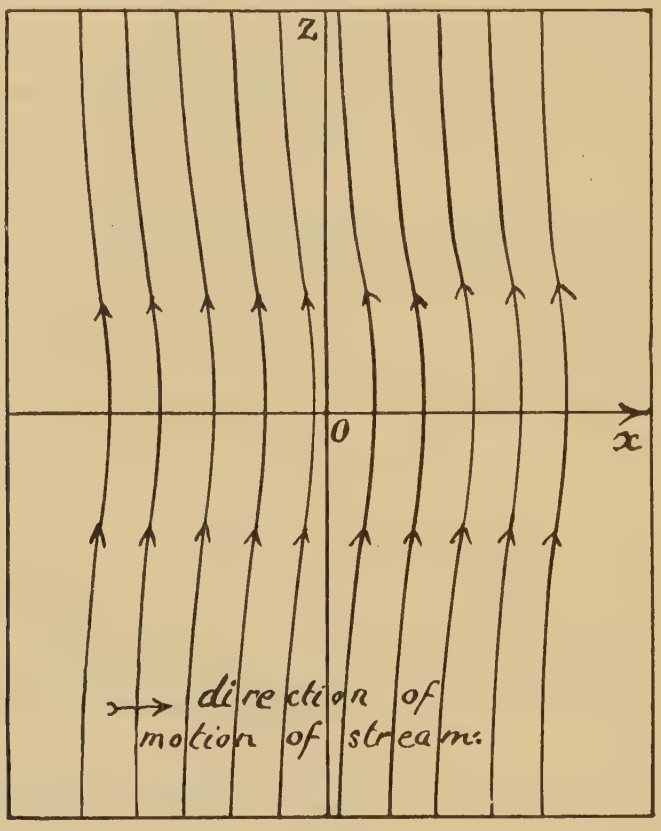

FIG. 1.

charges, so that the electric field is due entirely to space charge : the density of this distribution is found from Poisson's equation,

$$\rho = -\nabla^2 V/4\pi,$$

or, by (10), namely, $-\operatorname{div}(v_\wedge H)/4\pi$. From (16) and (17) we have

$$\rho = -(pv_0H_0/4\pi)\exp(-q\,|\,z\,|)\sin py. \tag{23}$$

Finally, the potential V in the interior of the stream is given by

$$V = v_0H_0\left(\frac{q\exp(-p\,|\,z\,|) - p\exp(-q\,|\,z\,|)}{q^2-p^2}\right)\sin py + \text{const.} \tag{24}$$

It is easily verified that if $q=0$, the solution reduces to that given in § 2 ; also if $H'=0$ everywhere, by (15), v is a function of y only ; this is equivalent to (1).

4. *Mechanical Forces Arising.*—The mechanical force per unit mass

resulting from the action of the magnetic field on the electric currents is equal to $\mathbf{J}_{\wedge}\mathbf{H}$; its components are therefore given by

$$\mathbf{F} = (\mathcal{J}_y H,\ \mathcal{J}_z H' - iH,\ -j_y H'). \qquad (25)$$

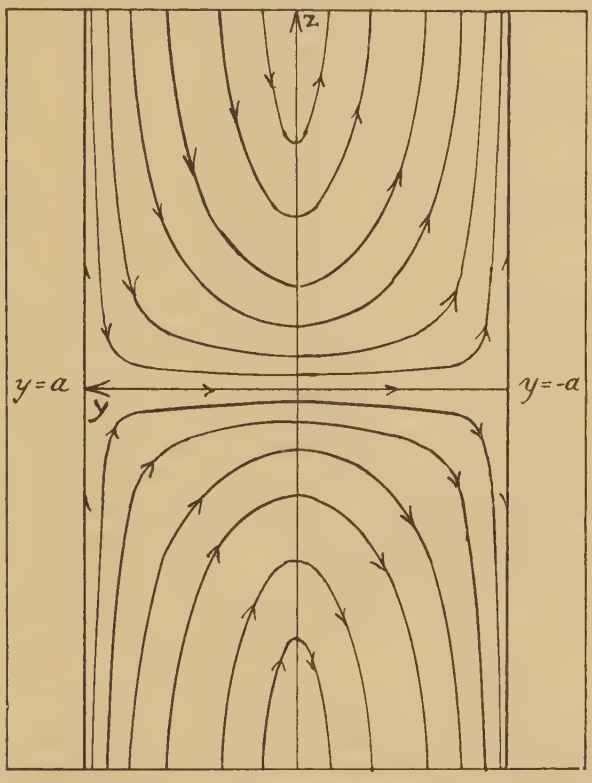

FIG. 2.

The x-component of $\mathbf{F}$ is negative, and its effect is therefore to oppose the motion of the stream in accordance with Lenz's law ; the y-component is positive in the region of the stream defined by $0 \leqslant y \leqslant a$, and negative in the remaining half, so that its effect is to distend the stream laterally. The z-component likewise is such as to distend the stream in opposite directions "above" and "below" the (x, y)-plane. Thus, the motion of the stream would only subsist unchanged if extraneous forces acted on the stream so as to balance the mechanical forces.

5. *The Rotational Problem.*—Consider an ionized gas, bounded by a surface in which every element revolves about a fixed axis ; in addition, let there be a circulation in meridian planes through the axis, the motion being the same in all such plane ; near the surface the particles in the gas will be supposed to make elastic impacts with the bounding surface. Let there be a magnetic field, symmetrical about the axis of rotation, but whose lines of force need not be in meridian planes. A steady state is supposed to have been attained.

The equations of the steady electro-magnetic field are *

$$\text{curl } \mathbf{H} = 4\pi \mathbf{J}, \tag{26}$$

$$\text{curl } \mathbf{E} = 0, \tag{27}$$

$$\text{div } \mathbf{H} = 0, \tag{28}$$

$$\text{div } \mathbf{E} = 4\pi\rho\ ; \tag{29}$$

(27) shows that $\mathbf{E}$ is derived from a potential V so that

$$\mathbf{E} = -\text{grad } V. \tag{30}$$

In addition we have the equation for the conduction of current

$$\mathbf{J} = \sigma(\mathbf{v}_\wedge \mathbf{H} - \text{grad } V) + \mathbf{i}\ ; \tag{31}$$

here i is the density of the current producing the imposed field, $\mathbf{v}$ the velocity of the gas at any point. If m denote the mass density of the gas, the equation of continuity is

$$\text{div } (m\mathbf{v}) = 0. \tag{32}$$

Choose a system of cylindrical polar co-ordinates (ϖ, ϕ, z) and let the z-axis be chosen as the axis of rotation ; on account of the axial symmetry of the problem, we have $\partial/\partial\phi \equiv 0$, and by (32) the circulation in meridian planes is derived from a Stoke's stream function, ψ, such that

$$mv_\varpi = -\frac{1}{\varpi}\frac{\partial\psi}{\partial z}, \qquad mv_z = \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}; \tag{33}$$

let further

$$v_\phi = \Omega\varpi, \tag{34}$$

then Ω denotes the angular velocity of the point considered, and is a function of ϖ and z only.

We can similarly analyse the magnetic field, $\mathbf{H}$, at any point into an azimuthal component, H', depending on ϖ and z only and a part $\mathbf{H}_s$, this vector lying entirely in meridian planes. By (28), $\mathbf{H}_s$ is associated with a function $U(\varpi, z)$, say, analogous to Stoke's stream function, such that

$$(H_s)_\varpi = -\frac{1}{\varpi}\frac{\partial U}{\partial z}, \qquad (H_s)_z = \frac{1}{\varpi}\frac{\partial U}{\partial\varpi}\ ; \tag{35}$$

in any meridian plane, $U = \text{const.}$ is the equation to a magnetic line of force of the field $\mathbf{H}_s$.

From (31) we have

$$\text{curl } (\mathbf{J}/\sigma) = \text{curl } (\mathbf{v}_\wedge \mathbf{H}) + \text{curl } (\mathbf{i}/\sigma),$$

and, using (26), this becomes

$$\text{curl}\left(\frac{1}{\sigma}\text{curl } \mathbf{H}\right) = 4\pi \text{ curl } (\mathbf{v}_\wedge \mathbf{H}) + 4\pi \text{ curl } (\mathbf{i}/\sigma)\ ; \tag{36}$$

this is the differential equation for H when σ, v, i are known at all points.

* All quantities are measured in e.m.u.

6. *Special Case when i is Wholly Azimuthal and H′ = 0.*—The electric meridianal currents, to which reference has been made in the introduction, would give rise to the component H' of the magnetic field: the condition that in the steady state such currents should not exist is that $H' = 0$ everywhere. Supposing that i possesses only an azimuthal component, we find from (33), (34), (35) that the ϕ-component of (36) is

$$\nabla^2 H' - \frac{H'}{\varpi^2} = -4\pi\sigma\left\{\frac{\partial(U,\Omega)}{\partial(\varpi, z)} + \frac{\partial}{\partial\varpi}\left(\frac{H'}{m\varpi}\frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{H'}{m\varpi}\frac{\partial\psi}{\partial\varpi}\right)\right\}$$
$$+\frac{1}{\sigma\varpi}\left\{\frac{\partial\varpi H'}{\partial\varpi}\cdot\frac{\partial\sigma}{\partial\varpi} + \frac{\partial\varpi H'}{\partial z}\cdot\frac{\partial\sigma}{\partial z}\right\}. \qquad (37)$$

If H' vanishes everywhere, it follows that

$$\frac{\partial(U,\Omega)}{\partial(\varpi, z)} = 0, \qquad (38)$$

so that, unless Ω is constant,

$$\Omega = f(U), \qquad (39)$$

f being an arbitrary function.

The meaning of this result is that Ω is constant over the surface traced out by the complete revolution of a line of force about the axis of rotation: this is the analogue of (1) in § 2 for the rotational problem.

When (39) is satisfied the electrostatic field resulting from the polarisation of the gas is found from (26) and (31); we have

$$\frac{\partial V}{\partial\varpi} = \Omega\frac{\partial U}{\partial\varpi}; \qquad \frac{\partial V}{\partial z} = \Omega\frac{\partial U}{\partial z},$$

there being no azimuthal component of E. These two equations can be combined to give

$$dV = \Omega dU,$$

and, using (39), this can be integrated to

$$V = \int^U f(x)dx + \text{const.} \qquad (40)$$

The space- and surface-charge distributions can then be deduced from ordinary potential theory: the equipotentials $V = \text{const.}$ are the same as the surfaces $U = \text{const.}$, verifying the fact that E and H_s are perpendicular vectors at any point; in the case of the Sun, the electric field due to the polarisation must be added to the ordinary Pannekoek-Rosseland field due to gravitation. This is the generalisation of Cowling's result.

As an illustration of the above, consider the case of an ionized spherical mass of gas rotating as a rigid body in a uniform magnetic field H, say, the axis of rotation being parallel to this field. From (35) we have

$$U = \tfrac{1}{2}H\varpi^2, \qquad (41)$$

and by (40),

$$V = \Omega U + \text{const.} = \tfrac{1}{2}\Omega H\varpi^2 + \text{const.} \qquad (42)$$

The density of space charge is given by

$$\rho = -\nabla^2 V/4\pi = -\Omega H/2\pi, \tag{43}$$

and so is constant; the surface-charge density is equal to

$$\frac{5}{12\pi} a\Omega H P_2(\cos\theta) + \text{const.};$$

here (a, θ, ϕ) are the spherical polar co-ordinates of any point on the surface. The arbitrary constant can be found in terms of the total charge on the gas.*

Taking $\Omega = 15°$ per day, and $H = 100\Gamma$, corresponding roughly to the conditions in the solar atmosphere, we find ρ to be of order $1{\cdot}6 \times 10^{-15}$ e.s.u.; this may be compared with the value $3{\cdot}6 \times 10^{-22}$ e.s.u. for the interior of the Sun arising from the Pannekoek-Rosseland field. It is at once seen that, on the surface layers at any rate, the rotation of the Sun in the presence of its magnetic field gives rise to much larger fields than do the gravitational forces: this was first pointed out by Cowling (*loc. cit.*, p. 152).

7. *The Rotational Problem: i Wholly Azimuthal, but* $H' \neq 0$. Suppose as before that i is purely azimuthal, but let us remove the restriction that $H' = 0$; H' itself satisfies (37). In this equation the first term on the right represents the effect of the electric meridianal currents, whilst the next two terms give the modification produced in H' by the meridianal mass-motion. The last two terms determine the distribution of the currents in the interior in cases when the conductivity σ is variable. Whereas the meridianal mass-motion may be important in connection with the radial limitation of the Sun's magnetic field,† it will be shown that it cannot affect appreciably the electric meridianal currents due to the non-uniform rotation. For this reason we shall not consider further the effect of the meridianal mass-motion in discussing (37), and shall accordingly set $\psi = 0$ in this equation, which now becomes

$$\nabla^2 H' - \frac{H'}{\varpi^2} - \frac{1}{\sigma\varpi}\left\{\frac{\partial \varpi H'}{\partial \varpi}\frac{\partial \sigma}{\partial \varpi} + \frac{\partial \varpi H'}{\partial z}\cdot\frac{\partial \sigma}{\partial z}\right\} = -4\pi\sigma\frac{\partial(U, \Omega)}{\partial(\varpi, z)}. \tag{44}$$

8. In attempting to obtain a solution of this equation giving some indication of the nature and of the order of magnitude of the component H' for the Sun, we require a knowledge of the solar magnetic field and of the variation of the angular velocity in the interior of the Sun. Our knowledge of both these is confined to observations of the solar atmosphere, and only extends into the interior for a small fraction of a solar radius.

On account of the symmetry of the solar mass-motion, both about the axis and with respect to the equatorial plane, Ω must be expressible in the form

$$\Omega = \Omega_0 + \Omega_1 \cos^2\theta + \Omega_2 \cos^4\theta + \ldots, \tag{45}$$

* *Cf.* Larmor, *Collected Works*, **1**, 8, 10, 1929.

† It was shown in an earlier paper (*M.N.*, **95**, 280, 1935) that in conjunction with the radial component of the magnetic field the meridian mass motion could in certain circumstances account for the radial limitation of the magnetic field.

θ being the heliocentric co-latitude, and Ω_0, Ω_1, . . . being functions of the radial distance, r, from the Sun's centre.

From theoretical considerations, Jeans * deduced that Ω_0, Ω_1, Ω_2, . . . vary like $1/r^2$, but we shall leave these functions undefined.

As regards the magnetic field, it will be sufficient for our purpose to assume that

$$U = Au(r) \sin^2 \theta, \tag{46}$$

where $u(r)$ is a non-dimensional function, corresponding to which the components of the magnetic field at the Sun's surface ($r = a$) vary with θ approximately in the observed way : it is, of course, not certain that this expression for U does well represent the facts either in the interior or even at the surface. With this expression for U the components of H_s are

$$(H_s)_r = -2A(u/r^2) \cos \theta, \tag{47}$$

$$(H_s)_\theta = (A/r)(du/dr) \sin \theta. \tag{48}$$

9. The electrical conductivity of stellar matter has been discussed by several authors : † it is found to increase from the star's surface downwards, its value at the centre being comparable with the conductivity of metals at ordinary temperatures. In the solar atmosphere the conductivity for current-flow in a direction at right angles to the magnetic field is likely to be reduced in value, but this will probably not be so in the interior.

We shall suppose that

$$\sigma = \sigma_0(r/a)^\beta, \tag{49}$$

where σ_0 and β are constants. If β is large and positive the currents will be confined to the surface layers ; and conversely, if β is large and negative, to a small core.

10. We shall now obtain a solution of (44) on the supposition that U and σ are given by (46) and (49) respectively, and that Ω is given by *the first two terms only* in (45) ; we impose the further condition that $u(r)$ shall be a function of r such that

$$u(a) = 0 \quad \text{and} \quad (du/dr)_{r=a} = 0; \tag{50}$$

this ensures that H_s vanishes at the boundary of the gas.

Transforming (44) to spherical polar co-ordinates (r, θ, ϕ), it becomes

$$\nabla^2 H' - \frac{H' \operatorname{cosec}^2 \theta}{r^2} - \frac{\beta}{r^2} \frac{\partial(rH')}{\partial r} = \frac{4\pi\sigma}{r} \frac{\partial(\Omega, U)}{\partial(r, \theta)} \tag{51}$$

$$= \frac{8\pi\sigma A}{r}(w_0 + w_1 \cos^2 \theta) \sin \theta \cos \theta,$$

where

$$w_0 = u\frac{d\Omega_0}{dr} + \Omega_1 \frac{du}{dr}, \tag{52}$$

$$w_1 = u\frac{d\Omega_1}{dr} - \Omega_1 \frac{du}{dr}. \tag{53}$$

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* J. H. Jeans, *Astronomy and Cosmogony*, ch. x, 1929.

† S. Chapman, *M.N.*, **89**, 54, 1928 ; T. G. Cowling, *M.N.*, **93**, 90, 1932.

It is convenient to write $H' = K \cdot g(r, \theta)$, where $g(r, \theta)$ is a non-dimensional function and

$$K = 8\pi\sigma_0 A, \tag{54}$$

and also

$$x = r/a\,; \tag{55}$$

this expresses the radial distances as a fraction of the radius of the boundary.

The equation (51) can now be written,

$$\nabla^2 g - \frac{g\,\mathrm{cosec}^2\,\theta}{x^2} - \frac{\beta}{x^2}\frac{\partial(xg)}{\partial x} = x^{\beta-1}(\omega_0 P_2^1(\cos\theta) + \omega_1 P_4^1(\cos\theta)), \tag{56}$$

where

$$\omega_0 = \tfrac{1}{3}w_0 + \tfrac{1}{7}w_1, \qquad \omega_1 = \tfrac{2}{35}w_1, \tag{57}$$

it being understood that in these expressions differentiation is now with respect to x.

The solution of (56) is therefore of the form

$$g(x, \theta) = g_0(x)P_2^1(\cos\theta) + g_1(x)P_4^1(\cos\theta), \tag{58}$$

where g_0 and g_1 satisfy respectively the equations

$$\frac{d}{dx}\left(x^2\frac{dg_0}{dx}\right) - \beta\frac{dg_0 x}{dx} - 6g_0 = x^{\beta+1}\omega_0, \tag{59}$$

$$\frac{d}{dx}\left(x^2\frac{dg_1}{dx}\right) - \beta\frac{dg_1 x}{dx} - 20g_1 = x^{\beta+1}\omega_1\,; \tag{60}$$

since $H' \propto g$, it follows from (58) that H' vanishes on the axis of rotation, over the equatorial plane, and that H' changes sign in crossing this plane: thus, as in the case of the "slab" problem (§ 3), the magnetic lines of force are bent forward in the direction of rotation.

Solutions have been found for various forms of the functions Ω_0 and Ω_1 (such that (38) is not satisfied, of course) and of the function u, subject to the condition (39); the formal solutions, which are involved and need not be given here, are found to possess the following characteristics :—

(*a*) The flow of electric meridianal currents (which is symmetrical in the four quadrants of any meridianal plane, and the same in all such planes) takes place throughout the sphere, unless β is very large and positive, when the flow is confined to a thin layer below the surface, or when β is very large and negative, when the flow is confined to a small core.

(*b*) The ratio H'/H_s is, by (47), (48) and (54), independent of A, and so of the absolute magnitude of H_s at any point.

(*c*) Taking $\sigma_0 = 10^{-9}$ e.m.u., which is a fair estimate of the conductivity in the solar atmosphere, calculations show that for the Sun, H'/H_s (except near the equator, where it is zero) is very large indeed except when β is large and positive,* or when the non-uniformity of the rotation is confined to a surface layer a few hundred kilometres thick.

* The ratio H'/H_s would be of order unity if β is such that the current distribution is practically confined to a surface layer about a hundred kilometres thick.

Similar conclusions may be more simply deduced from the problem considered in § 3 ; as far as orders of magnitude are concerned we may take the thickness of the slab to be equal to the Sun's diameter ; and taking also $p=q$ (this giving an appreciable decrease in v for a distance comparable with a solar radius), we find from (18*a*) that the order of magnitude of the ratio H'/H is $-2\pi\sigma v z$; setting $\sigma=10^{-9}$ e.m.u. as before, $v=2\cdot10^5$ cm./sec. (the Sun's equatorial velocity), the value of this ratio becomes of the order of $(6\cdot3\times10^{-5})z$, so that at a point distant one-half a solar radius on either side of the plane $z=0$, H'/H would be as high as $4\cdot10^6$. But if $p>>q$, corresponding to the case when the cross-currents only flow within a thin layer of the Sun's atmosphere, we find that, except near the plane $z=0$, H'/H is of the order of $-4\pi\sigma v q\delta/p^2$; taking the same values of σ and v as before and $qa=1$, and the thickness of the slab only 100 km., we find H'/H to be about 1·5 ; for thinner layers the ratio would be still less.

11. As we shall now see, it is highly improbable that such a flow of electric currents as has been described above can take place in the Sun.

Consider firstly the mechanical forces arising : as in the problem considered in § 3, there will be three components of the mechanical force of which the azimuthal component will be such as to slow down the rotation (*cf.* § 4). It will be sufficient for our purpose to consider solely this component, and to use the simpler results deduced in §§ 3 and 4 ; from (25) the x-component of the mechanical form is equal to $\mathcal{J}_y H$: this is greatest in the plane $z=0$, where its value is $\sigma v_0 H_0^2(q/(p+q))\cos^2 py$. Thus we may estimate the retarding force as of the order of $\{q/(p+q)\}\sigma v_0 H_0^2$: in applying this result to estimate the force slowing down the solar rotation, v_0 should not be taken to be the full equatorial velocity of the Sun ; a better procedure is to take v_0 equal to the velocity at the equator corresponding to the differential angular velocity between the equator and the poles. This gives for v_0 a value of the order of 10^4 cm./sec. ; taking also $\sigma=10^{-9}$ e.m.u., $p=q$ and $H_0=100\Gamma$, the retarding force per unit volume is found to be of the order of 0·05 dynes/c.c. The effect of this mechanical action is to destroy the non-uniformity of the rotation, tending to make it conform to the law given by (39) ; the ordinary viscous drag also tends to render uniform the angular velocity of the Sun, and in the first instance it is of interest to compare with this drag the retarding force due to the magnetic action. The viscous drag per unit volume may be estimated to be of the order of $\mu v/a^2$, v being the velocity in any latitude, a the radius of the Sun and μ the coefficient of viscosity of the solar atmosphere ; μ is unlikely to exceed the value 100 c.g.s. Taking $v=2\cdot10^5$ cm./sec. the viscous drag is found to be of the order of $4\cdot10^{-15}$ dynes/c.c., and is thus negligible in comparison with the electromagnetic retarding force, f_m say.

This shows that the latter is far more effective than the viscous drag in rendering the angular velocity of the Sun uniform ; it becomes necessary to compare the electromagnetic force with the forces which sustain the non-uniform rotation. On Jeans's theory * this is attributed to the braking action of radiation in passing outwards through the successive layers of the

* Jeans, *Astronomy and Cosmogony*, pp. 266–275, 1928.

Sun. If $\mathcal{H}$ denotes the flux of radiant energy per unit area radially outwards and c the velocity of light, the contribution to the angular momentum of an element of unit volume per second is $d(\mathcal{H}\Omega r^4/c^2)/dr$; the contribution from the mechanical action of the electric currents in the presence of the magnetic field is of the order of $\sigma\Omega H r^4$ per second. Hence, as far as orders of magnitude are concerned, the ratio of the former expression to the latter may be estimated as

$$\mathcal{H}/Hr\sigma c^2. \tag{61}$$

The value of $\mathcal{H}$ at the surface is $6{\cdot}24 \cdot 10^{10}$ ergs per second, and its value in the interior, though much larger, is unlikely to exceed the value at the surface by a factor greater than 10^4; * as H and σ also increase inwards, the value of the ratio (61) is unlikely to exceed the value at the surface by a factor greater than 10^3, say. Taking $r = 7{\cdot}10^{10}$ cm., $H = 100\Gamma$, $\sigma = 10^{-9}$ e.m.u., $c = 3{\cdot}10^{10}$ cm./sec., the value of this ratio at the surface may be estimated at 10^{-10} at the most. Ignoring the viscous drag in comparison with the electromagnetic retarding force, it is easily shown that the latter would be effective in destroying any non-uniformity in the solar rotation or make it conform to the law given by (39). The time taken for this adjustment is small compared with the age of the Sun.

12. It may be urged that since the magnetic field does not act on the neutral atoms, the charge particles might conform to a rotation which is either uniform or obeys the law given by (39), whilst the neutral atoms would be free to move in any manner. This is unlikely to be the case; for the diffusion of the ions and electrons through the neutral atoms would give rise to a reaction between the two streams tending to destroy their differential motion. The magnitude of this reaction P may be estimated from formulæ developed by McCrea; † these are of two types according as the differential velocity, U, of the two streams is or is not large in comparison with the mean thermal velocities. The formulæ are

$$P = \tfrac{8}{3}\nu_1\nu_2(d_1+d_2)^2\left\{2\pi kT\frac{m_1m_2}{m_1+m_2}\right\}^{\frac{1}{2}}U,$$

and

$$P = \pi\nu_1\nu_2(d_1+d_2)^2\frac{m_1m_2}{m_1+m_2}U^2.$$

In our case ν_1, ν_2 are respectively the number densities of the ions (and electrons) and of the neutral atoms; (d_1+d_2) is the effective cross-section of collision, T the temperature, and k Boltzmann's constant. Taking $\nu_1 = 5{\cdot}10^{13}$ ions/c.c., $\nu_2 = 5{\cdot}10^{15}$ atoms/c.c., $T = 6000°$, these values corresponding roughly to the conditions in the lower reversing layer, $d_1+d_2 = 10^{-8}$ cm., $m_1 = 10^{-23}$ gm., $m_2 = 10^{-24}$ gm. (to allow for the preponderance of hydrogen), the above formulæ become respectively

$$P = 1{\cdot}5 \times 10^{-4}U \text{ dynes/c.c.} \tag{62a}$$

and

$$P = 7{\cdot}8 \times 10^{-11}U^2 \text{ dynes/c.c.} \tag{62b}$$

* *Cf.* Eddington, *The Internal Constitution of the Stars*, p. 114, 1926.

† *M.N.*, **95**, 509, 1935.

As an example, we may suppose the differential velocity U of the atoms and ions (and electrons) to correspond to the velocity at the equator of a particle moving with the differential angular velocity of the Sun between the equator and the poles; this would give U of the order of 10^4 cm./sec., which is small compared with the thermal velocities of the atoms of the order of 10^5 cm./sec.; substituting $U = 10^4$ in (62*a*) and $U = 10^5$ in (62*b*), the values of P are found to be of order unity and so very large compared with the radiation force which, on Jeans's theory, produces the anomalous rotation. Hence, if a differential velocity exists between the ions and neutral atoms it must be very small. For instance, in order that the force P and the radiation force may be comparable, U would only be of the order of 10^{-7} cm./sec.; they would thus move practically together.

13. Finally, we consider the possibility that the electric currents flowing in meridian planes, giving rise to H', might not be considerably checked by a circulation of the atmosphere in meridian planes (such as has been supposed to account for the radial limitation) taking place from the poles to the equator and completed below the surface. To estimate the magnitude of this effect it will suffice to consider the problem of § 3 in which we shall now suppose that there exists a circulation in planes parallel to the (y, z)-plane, the flow being the same in all such planes and like the flow of electric currents depicted in fig. 2. Let (u, w) be the x- and z-components of the velocity of this circulation in any plane $x = \text{const.}$; w will then be zero in the plane $z = 0$ and will change sign with z. The y-component of the force per unit charge acting on an ion or electron is the only one with which we are concerned: this is given by

$$E_y = wH' - vH - \partial V/\partial y. \tag{63}$$

Since the current density is proportional to E_y, the condition that H' be very small is that E_y vanishes very nearly. (It is to be noted that H' cannot be made to vanish exactly for any prescribed circulation.)

We shall use the results of § 3 to estimate the order of magnitude of the component of the electric field, $-\partial V/\partial y$, in (63): since v diminishes very rapidly for values of $|z|$ such that $q|z| > 1$ (if (17)), we need not consider regions beyond those defined by this inequality.

For reasons already stated, we may also exclude values of p and q differing greatly from each other. Accordingly, taking $p = q$ from (13) and (18) we find that the last two terms on the right of (63) are of the order of $-\frac{1}{2}vH(1 - q|z|)$: thus we may expect that, as far as order of magnitude is concerned, E_y is approximately given by

$$E_y = wH' - fvH, \tag{64}$$

where $f(< 1)$ is of order unity.

For E_y to be small, H'/H must very nearly equal fv/w, whence it follows that if $H'/H << 1$, then $fv/w << 1$, or since f is of order unity, $v/w << 1$. This would mean that the observed velocity of the solar atmosphere due to the rotation should be small in comparison with any meridianal velocities which it may possess. Since this is certainly not the case, we conclude that

any small circulation in meridian planes (such as could account for the radial limitation of the magnetic field) is of no interest in the present connection.

14. *Conclusion.*—The results of the last two sections suggest that the magnetic field is likely to be an important factor in any consideration of the non-uniform rotation of the Sun. If, as seems not unlikely, the magnetic field emanates from the deep interior, then it seems probable that the angular velocity of the Sun, if it be not uniform throughout, must adjust itself so as to satisfy to a high degree of approximation the law $\Omega = f(U)$ deduced in § 6. As has been said before, this means that Ω must be constant over the surfaces traced out by the rotation of the magnetic lines of force about the axis. Without more definite observational data as regards the variation of Ω and U with depth in the solar atmosphere, it seems difficult to make any definite deduction from this law. But it suggests at least that the increase of the magnetic field with depth is unlikely to be so rapid as has been previously supposed, since, if the magnetic lines of force are indeed very nearly parallel to the Sun's surface, then Ω should be practically constant over this surface ; it is an observed fact that Ω is not constant. Thus we must suppose that the magnetic lines of force spread out more into the space surrounding the Sun instead of being confined almost entirely within the layers below the chromosphere.

The work of Evershed * and St. John † on the increase of the angular velocity with height in the chromosphere also lends support to this suggestion. Their data suggest that the inclination to the spherical surface of the Sun of the surfaces $\Omega = \text{const.}$ lies in a range of approximately 1° to 10°. The law $\Omega = f(U)$ then implies a similar range for the inclination to the same spherical surface of the magnetic lines of force, and not the small fraction of a degree that is deduced on the supposition that the magnetic field is considerably reduced in the thousand kilometres or so of the solar atmosphere. Further data are necessary to settle this point, especially if one remembers that the *increase* observed by Evershed and St. John takes place in the chromosphere and not in the reversing layer.

Note added 1937 *April* 12.

If the magnetic field of the Sun is not limited as strongly as has been supposed, it will not vanish at the surface, and hence surface charges will also be induced ; but, as it has been mentioned in the introduction, these would rapidly disperse into the outer space under the mutual repulsion of charges of like signs ; it seems likely that the density of these charges will therefore be small and that the charges will tend to "hook" themselves on to the magnetic lines of force. The flow of charged particles would then be almost entirely along the lines of force.

* *M.N.*, **85**, 609, 1925.
† *Mt. Wilson Contributions*, No. 278, 1924.

STELLAR KINEMATICS AND MEAN PARALLAXES.

Frank K. Edmondson.

(Communicated by Dr. Harlow Shapley)

Introduction

1. Generalizing an analysis first published by Ogrodnikoff,* E. A. Milne has demonstrated † on the basis of purely kinematic considerations that "any spatio-velocity stellar-distribution function whatever which possesses certain differential coefficients is accompanied by, in general, a K-term and a double wave in radial velocities, and a systematic drift and double wave in the transverse motions in any arbitrary plane." This led him to the conclusion that "the existence of a double wave in the radial velocities is *by itself* no evidence for galactic rotation, though this is not to deny the reality of galactic rotation on other evidence."

Furthermore, Milne pointed out that his results are independent of the occurrence of star-streaming, since "star-streaming expresses *dispersion* features of the velocity distribution, whilst we have only found it necessary to discuss mean values." However, the existence of asymmetry, which is a correlation between velocity dispersion and mean group velocity, causes great uncertainty in the dynamical interpretation of the results of such a kinematic analysis. For example, if the percentage of high-velocity stars varies systematically with distance from the galactic centre, the sum of Oort's constants will not give the derivative of the circular velocity with respect to R. As a result, the inferred law of force in the galaxy will differ from the true law of force.

Finally, a point of considerable theoretical and practical importance in galactic rotation studies is Milne's demonstration that "when applying corrections for 'solar motion,' in order that the results may have simple physical interpretations, it is necessary to use the solar motion as given by the stars *immediately* surrounding the Sun, and not that derived from a reduction of the motions of the stars under consideration." ‡

2. The present paper is an extension of Milne's work to second order terms. Following his notation, we take X, Y and Z to be co-ordinates with respect to an arbitrary origin O, and U, V and W to be the components of velocity in the X, Y and Z directions. S_1^0, S_2^0 and S_3^0 are the components

* "A Theory of Streaming in the System of B Stars," *Z. f. A.*, **4**, 190–207, 1932.

† "Stellar Kinematics and the K-effect," *M.N.*, **95**, 560, 1935.

‡ The existence of asymmetry makes it necessary to modify this statement to read ". . ., it is necessary to use the solar motion as given by the stars of moderate velocity dispersion immediately surrounding the Sun, . . ." It should also be mentioned here that one should not expect to find a normal galactic rotation effect in the velocities of a group of stars which yield an abnormal solar motion, since an abnormal solar motion is evidence of a significant deviation from circular motion (*i.e.* deviation from the necessary condition for a normal galactic rotation effect).

of the "local solar motion," and r is the distance from the Sun. x, y and z are co-ordinates with respect to the Sun as origin. Except for replacing Milne's equation (6) by

$$\bar{U} - U_0 = x\left(\frac{\partial \bar{U}}{\partial X}\right)_0 + y\left(\frac{\partial \bar{U}}{\partial Y}\right)_0 + z\left(\frac{\partial \bar{U}}{\partial Z}\right)_0 + \tfrac{1}{2}\left[x^2\left(\frac{\partial^2 \bar{U}}{\partial X^2}\right)_0 + y^2\left(\frac{\partial^2 \bar{U}}{\partial Y^2}\right)_0 + z^2\left(\frac{\partial^2 \bar{U}}{\partial Z^2}\right)_0\right] + xy\left(\frac{\partial^2 \bar{U}}{\partial X \partial Y}\right)_0 + xz\left(\frac{\partial^2 \bar{U}}{\partial X \partial Z}\right)_0 + yz\left(\frac{\partial^2 \bar{U}}{\partial Y \partial Z}\right)_0 - S_1^0$$

we have followed his procedure in detail, and the reader is referred to his paper for the steps followed in deriving the results of the present paper.

Corresponding to Milne's equations (12) and (18) we find that, in general, the radial velocity and the components of the tangential velocity in latitude and longitude (all corrected for "local solar motion") are of the form

$$a_0 + a_1 \cos\lambda + a_2 \cos 2\lambda + a_3 \cos 3\lambda + b_1 \sin\lambda + b_2 \sin 2\lambda + b_3 \sin 3\lambda. \quad (1)$$

The coefficients a_i, b_i are functions of r, β, and the first and second partial derivatives of $\bar{U}$, $\bar{V}$ and $\bar{W}$ with respect to X, Y and Z at the Sun. The general expressions are given in Appendix I.

The Solar Motion

3. It has already been pointed out by Miss Hayford * that the first harmonics due to galactic rotation combine with the first harmonic due to solar motion, the result being that it is impossible to separate the two effects in practice. Equation (1) generalizes this result, thereby giving added significance to the distinction between "local solar motion" and the solar motion as ordinarily defined.

4. The solar motion as ordinarily defined is obtained from radial velocities and proper motions by means of a least squares solution, the equations of condition being

$$\bar{\rho}(\lambda\beta) = -X_0 \cos\lambda \cos\beta - Y_0 \sin\lambda \cos\beta - Z_0 \sin\beta + K,$$

$$4{\cdot}74\bar{\mu}_\lambda''(\lambda\beta) = \frac{X_0}{r}\sin\lambda - \frac{Y_0}{r}\cos\lambda,$$

$$4{\cdot}74\bar{\mu}_\beta''(\lambda\beta) = \frac{X_0}{r}\cos\lambda \sin\beta + \frac{Y_0}{r}\sin\lambda \sin\beta - \frac{Z_0}{r}\cos\beta.$$

For the stars immediately surrounding the Sun, X_0, Y_0 and Z_0 are identical with S_1^0, S_2^0 and S_3^0 if we exclude moving clusters and other objects showing systematic deviations from the mean motion. For distant stars the terms in $\sin\lambda$ and $\cos\lambda$ and the constant term in (1) will cause X_0, Y_0 and Z_0 to differ from S_1^0, S_2^0 and S_3^0 by an amount depending on r and β.

In the case of the radial velocity, both the first and second order components of a_1 and b_1 contain $\cos\beta$ as a factor. The first order component of a_0 is the K-term, and the second order component contains $\sin\beta$ as a factor. Hence, by appropriately averaging the remaining part of each term

* *L.O.B.*, 448, 1932.

over the sky (*i.e.* by taking galactic concentration into account), one sees that the three elements of the solar motion (L, B and V_0) are, in general, functions of r. If, instead of averaging over the entire sky, we treat each zone of latitude separately, it is clear that L, B and V_0 are functions of r and β. A comparison of the coefficients for tangential velocity in longitude and latitude with the above equations of condition for proper motion in longitude and latitude leads to similar conclusions.

Detailed examination of the coefficients in Appendix I will show that the apex derived from radial velocities will not agree with the apex derived from proper motions. For example, the three values of X_0 are :

From Radial Velocities

$$S_1^0 - r\left(\frac{\partial \bar{U}}{\partial Z} + \frac{\partial \bar{W}}{\partial X}\right)_0 \sin\beta - r^2\left\{\frac{1}{2}\left(\frac{\partial^2 \bar{U}}{\partial Z^2} + 2\frac{\partial^2 \bar{W}}{\partial X \partial Z}\right)_0 \sin^2\beta + \frac{1}{8}\left(3\frac{\partial^2 \bar{U}}{\partial X^2} + \frac{\partial^2 \bar{U}}{\partial Y^2} + 2\frac{\partial^2 \bar{V}}{\partial X \partial Y}\right)_0 \cos^2\beta\right\}.$$

From Tangential Velocities in Longitude

$$S_1^0 - r\left(\frac{\partial \bar{U}}{\partial Z}\right)_0 \sin\beta - r^2\left\{\frac{1}{2}\left(\frac{\partial^2 \bar{U}}{\partial Z^2}\right)_0 \sin^2\beta + \frac{1}{8}\left(\frac{\partial^2 \bar{U}}{\partial X^2} + 3\frac{\partial^2 \bar{U}}{\partial Y^2} - 2\frac{\partial^2 \bar{V}}{\partial X \partial Y}\right)_0 \cos^2\beta\right\}.$$

From Tangential Velocities in Latitude

$$S_1^0 - r\left\{\left(\frac{\partial \bar{U}}{\partial Z}\right)_0 \sin^2\beta - \left(\frac{\partial \bar{W}}{\partial X}\right)_0 \cos^2\beta\right\} \csc\beta - r^2\left\{\frac{1}{2}\left(\frac{\partial^2 \bar{U}}{\partial Z^2}\right)_0 \sin^2\beta + \frac{1}{8}\left(3\frac{\partial^2 \bar{U}}{\partial X^2} + \frac{\partial^2 \bar{U}}{\partial Y^2} + 2\frac{\partial^2 \bar{V}}{\partial X \partial Y} - 8\frac{\partial^2 \bar{W}}{\partial X \partial Z}\right)_0 \cos^2\beta\right\}.$$

5. If we now introduce the hypothesis of galactic rotation, the formulæ become much simpler, and we can give an estimate of the size of the effect to be expected. Assuming that $\bar{U}$ and $\bar{V}$ are independent of Z and $\bar{W} = 0$ everywhere,* we obtain the formulæ in Appendix II after transformation to cylindrical co-ordinates referred to the galactic centre. Following Milne's notation, we have put L = velocity along the radius and M = velocity of rotation.† The longitude λ is now measured from the direction to the galactic centre. Examining the effect on the solar motion as before we find that X_0 and Y_0 derived from radial velocities and proper motions in latitude will agree, but that different values will be obtained from proper motions in longitude. The value of Z_0 derived from radial velocities will differ from that derived from proper motions in latitude.

If we introduce the final assumptions of galactic rotation ($\bar{L} = 0$, and $\bar{M}$

* A north-south asymmetry in mean secular parallaxes, and in the solar motion, would indicate that these assumptions are not correct.

† Milne took rotation to be positive in the counter-clockwise direction. Following Oort and others we have taken rotation to be positive in the clockwise direction. This introduces a few differences in sign between Milne's results and ours.

independent of θ), the coefficients in Appendix II become equivalent to those given by Miss Hayford.* In this case we have :

Radial Velocity

$$a_i = 0, \qquad b_1 = r^2 Q_1 \cos^3\beta, \qquad b_2 = rA\cos^2\beta, \qquad b_3 = r^2 Q_2 \cos^3\beta.$$

Tangential Velocity in Longitude

$$a_0 = rB\cos\beta, \quad a_1 = 3r^2 Q_1 \cos^2\beta, \quad a_2 = rA\cos\beta, \quad a_3 = r^2 Q_2 \cos^2\beta, \quad b_i = 0.$$

Tangential Velocity in Latitude

$$a_i = 0, \qquad b_1 = -r^2 Q_1 \sin\beta\cos^2\beta, \qquad b_2 = -rA\sin\beta\cos\beta,$$
$$b_3 = -r^2 Q_2 \sin\beta\cos^2\beta,$$

where

$$Q_1 = \tfrac{1}{8}\left[\frac{d^2\bar{M}}{dR^2} + \frac{1}{R}\frac{d\bar{M}}{dR} - \frac{\bar{M}}{R^2}\right]$$

$$Q_2 = \tfrac{1}{8}\left[\frac{d^2\bar{M}}{dR^2} - \frac{3}{R}\frac{d\bar{M}}{dR} + \frac{3\bar{M}}{R^2}\right]$$

and A and B are Oort's coefficients.

6. The observed values of Oort's coefficients indicate that in the neighbourhood of the Sun $\dfrac{d\bar{M}}{dR} = 0$, and $\dfrac{\bar{M}}{R} = \cdot 030$. If $\bar{M}$ is taken to be 300 km./sec., then $R = 10{,}000$ parsecs. Hence, we have

$$Q_1 = \tfrac{1}{8}\left[\frac{d^2\bar{M}}{dR^2} - 3\cdot 10^{-6}\right],$$

$$Q_2 = \tfrac{1}{8}\left[\frac{d^2\bar{M}}{dR^2} + 9\cdot 10^{-6}\right].$$

7. Confining our attention only to those terms that affect the solar motion as ordinarily defined, we obtain

$$\bar{\rho} = (-S_1{}^0)\cos\lambda\cos\beta + (-S_2{}^0 + r^2 Q_1\cos^2\beta)\sin\lambda\cos\beta + (-S_3{}^0)\sin\beta,$$

$$4\cdot 74\,\bar{\mu}_\lambda'' = \left(+\frac{S_1{}^0}{r}\right)\sin\lambda + \left(-\frac{S_2{}^0}{r} + 3rQ_1\cos^2\beta\right)\cos\lambda,$$

$$4\cdot 74\,\bar{\mu}_\beta'' = \left(+\frac{S_1{}^0}{r}\right)\cos\lambda\sin\beta + \left(+\frac{S_2{}^0}{r} - rQ_1\cos^2\beta\right)\sin\lambda\sin\beta + \left(-\frac{S_3{}^0}{r}\right)\cos\beta.$$

The longitude λ is measured from the direction to the galactic centre.

It is clear that our previous conclusions are not essentially modified by introducing the restricted case of galactic rotation. L, B, and V_0 are still functions of r and β, and the apex derived from radial velocities and proper motions in latitude will not agree with the apex derived from proper motions

* *L.O.B.*, 448, 1932.

in longitude. These results have an mportant bearing on the problem of determining mean parallaxes from proper motions. We shall derive the effects to be expected for the two most used methods, secular parallaxes and tau-components.

Secular Parallaxes

8. In many investigations the position of the apex is assumed to be known and mean secular parallaxes are computed by means of the formula

$$\left(\frac{\bar{h}}{\bar{r}}\right)=\frac{\overline{v \sin \Delta}}{\overline{\sin^2 \Delta}},$$

where Δ is the distance from the assumed apex and v is the component of the proper motion directed away from the apex. The mean secular parallax is related to the mean annual parallax by the equation

$$\left(\frac{\bar{h}}{\bar{r}}\right)=\left(\frac{\overline{V_0}}{4{\cdot}74r}\right)=\frac{V_0}{4{\cdot}74}\bar{\pi}.$$

9. The results stated in Part 7 indicate that, for very distant stars, the relationship between mean secular parallax and mean annual parallax will be more complicated. It is not difficult to show that the mean secular parallax in a zone of latitude is given by

$$\left(\frac{\bar{h}}{\bar{r}}\right)=\frac{V_0}{4{\cdot}74}\bar{\pi}+\frac{f(\beta)Q_1}{4{\cdot}74}\bar{r},$$

where

$$f(\beta)=\frac{\sin L \cos B \cos^2 \beta(\sin^2 \beta+3)}{\cos^2 B \cos^2 \beta+2(\sin^2 B \sin^2 \beta-1)}.$$

To prove this, μ_λ and μ_β from section 7 are projected on the v direction. From the resulting equation an expression for $v \sin \Delta$ is obtained and is averaged with respect to λ for a zone of latitude. The formula for $\sin^2 \Delta$ is also averaged with respect to λ for a zone of latitude. The quotient of these two expressions gives the above formula for the mean secular parallax. The first term on the right is exact only if L and B are the co-ordinates of the "local solar apex" and V_0 is the "local solar velocity." However, the formula is applicable in practice even if the assumed apex does not coincide with the "local solar apex." *

If we put $L=60°$, $B=20°$, the following values of $f(\beta)$ are obtained :—

β	$f(\beta)$	β	$f(\beta)$
0°	−2·20	45°	−1·00
15°	−2·01	60°	−0·48
30°	−1·56	90°	0·00

* Strömberg, "Computation of Mean Parallaxes, etc.," *Mt. Wilson Contr.*, 558; *Ap. J.*, 84, 560, 1936.

Hence, if Q_1 is negative (as seems probable), the mean secular parallax will be large for nearby stars and for distant stars, and will reach a minimum at an intermediate distance.

If we denote by π_v the mean annual parallax derived from the mean secular parallax in the ordinary way, we have

$$\pi_v = \frac{4{\cdot}74}{V_0}\left(\frac{\bar{h}}{\bar{r}}\right) = \bar{\pi} + f(\beta)\frac{Q_1}{V_0}\bar{r}.$$

If we put (see Part 14)

$$\bar{\pi} \,.\, \bar{r} = K(\beta) \geqslant 1,$$

then

$$\pi_v = \frac{K(\beta)}{\bar{r}} + \frac{Q_1}{V_0} f(\beta)\bar{r}$$

$$= \bar{\pi} + \frac{Q_1}{V_0}\frac{f(\beta)K(\beta)}{\bar{\pi}}.$$

It is not possible to predict the value of $(\pi_v - \bar{\pi})$, since the value of Q_1 involves the unknown second derivative of the rotational velocity with respect to the distance from the galactic centre. However, it is worth while to give some figures to show the order of magnitude of the effect that may exist. If $\frac{d^2\overline{M}}{dR^2} = 0$, the effect does not become appreciable for distances less than 2000 parsecs, and at a distance of 3000 parsecs π_v is approximately twice $\bar{\pi}$ for latitudes less than 30°. If we assume $\frac{d^2\overline{M}}{dR^2} = -3{\cdot}10^{-5}$, the effect is appreciable at 2000 parsecs, as the following table shows :—

Hypothetical Values of π_v

β \ $\bar{\pi}$	0″·0010	0″·0005	0″·0003
°			
0	0″·0020	0″·0024	0″·0032
15	19	22	29
30	17	18	23
60	12	8	8
90	10	5	3

Since

$$\pi_v(MIN) = 2\sqrt{\frac{Q_1}{V_0}f(\beta)K(\beta)}$$

the minimum value of π_v decreases with increasing latitude.

Tau-Components

10. Since the tau-component is defined as the component of the proper motion at right angles to the direction of the apex, it follows that if the solar motion is the only systematic influence at work the mean tau-component

should be zero. The second order galactic rotation terms introduce the effect

$$\bar{\tau}_{\lambda\beta} = \left| \frac{rQ_1}{4{\cdot}74} \frac{\cos^2\beta(A\cos\lambda + B\sin\lambda\cos\lambda + C\cos^2\lambda + D)}{\sqrt{a\sin\lambda + b\cos\lambda + c\sin\lambda\cos\lambda + d\sin^2\lambda + e\cos^2\lambda + f}} \right|$$

where

$$A = 3\sin B\cos\beta, \qquad B = -2\sin L\cos B\sin\beta,$$
$$C = -2\cos L\cos B\sin\beta, \qquad D = C/2,$$

$$a = -2\sin L\sin B\cos B\sin\beta\cos\beta, \qquad b = -2\cos L\sin B\cos B\sin\beta\cos\beta,$$
$$c = -2\sin L\cos L\cos^2 B\cos^2\beta, \qquad d = -\sin^2 L\cos^2 B\cos^2\beta,$$
$$e = -\cos^2 L\cos^2 B\cos^2\beta, \qquad f = 1 - \sin^2 B\sin^2\beta.$$

11. If the mean tau-component is zero, and if the individual values are distributed according to a normal error curve with dispersion σ, then

$$|\bar{\tau}| = \sqrt{\frac{2}{\pi}}\sigma.$$

If the mean is different from zero, then

$$|\bar{\tau}| = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\left\{\tau_0\int_0^{\tau_0} e^{-\frac{\tau^2}{2\sigma^2}}d\tau - \int_0^{\tau_0}\tau e^{-\frac{\tau^2}{2\sigma^2}}d\tau + \sigma^2\right\},$$

where τ_0 is the mean.

Since the mean annual parallax is proportional to the mean of the absolute values of the tau-components, it follows that the computed mean parallax will always be too large by the factor :

$$k = \frac{1}{\sigma^2}\left\{\bar{\tau}_{\lambda\beta}\int_0^{\bar{\tau}_{\lambda\beta}} e^{-\frac{\tau^2}{2\sigma^2}}d\tau - \int_0^{\bar{\tau}_{\lambda\beta}}\tau e^{-\frac{\tau^2}{2\sigma^2}}d\tau + \sigma^2\right\}.$$

A brief table of some values of k is given in Table I.

TABLE I

$\frac{\bar{\tau}_{\lambda\beta}}{\sigma}$	k
0·00	1·00
0·50	1·12
1·00	1·46
1·50	1·95
2·00	2·53

12. If $\frac{d^2\bar{M}}{dR^2} = -3{\cdot}10^{-5}$, the mean value of k at 3000 parsecs will be of the order of 1·5. If the apex were in the galactic plane, k would be 1·0 in all longitudes for stars at latitude zero. The exaggeration of π_v in low latitudes is by a factor of 10 for the same conditions. Hence, if the conditions assumed

in Part 5 exist, τ-components will be more reliable than v-components. Furthermore, since the effect on τ-components is caused by $\bar{\tau} \neq 0$, it is possible to eliminate it entirely by measuring the τ-components from their mean.

The Higher Harmonics

13. If Q_1 has been found from its effect on mean parallaxes, then theoretically we can predict that there will be third harmonic terms in radial velocities and proper motions with amplitude

$$Q_2 = Q_1 + 1 \cdot 5 \cdot 10^{-6}.$$

It is to be doubted, however, whether any *observed* third and higher harmonics in radial velocities and proper motions of distant objects as a function of longitude have any kinematic or dynamical significance, because irregularities in the absorbing material in the galaxy will cause the mean distance to be a harmonic function of longitude, thus affecting the other harmonics.

As a simple illustration, suppose the material is distributed in such a way that the mean distance is related to the longitude by the formula

$$r = r_0(1 - \epsilon \cos \lambda) \qquad \epsilon < 1.$$

Then the galactic rotation term in the radial velocities will be

$$\begin{aligned} rA \sin 2\lambda &= r_0 A \sin 2\lambda - \epsilon r_0 A \cos \lambda \sin 2\lambda \\ &= r_0 A \sin 2\lambda - \epsilon r_0 A/2 \,(\sin 3\lambda + \sin \lambda), \end{aligned}$$

thus introducing spurious first and third harmonics. Since the absorption is certainly a more complicated function of longitude than is implied by this expression, the harmonics actually observed will be an uncertain mixture of absorption and second order terms in the velocities.

Similarly, the solar motion terms in the proper motions will introduce spurious second and higher harmonics, thereby falsifying the Oort first order coefficients as well as the second order terms. Hence we conclude that the third harmonic cannot be used to check the value of Q_1.

Relationship between $\bar{\pi}$ and $\bar{r}$

14. It is well known that while $r = 1/\pi$, $\bar{r} \neq 1/\bar{\pi}$. This is because $\bar{r}$ is the arithmetic mean distance and $1/\bar{\pi}$ is the harmonic mean distance. It can be proved that the harmonic mean of a series of numbers is always less than, or at most equal to (if all the numbers are the same), the arithmetic mean. This is equivalent to the statement that

$$\bar{\pi} \cdot \bar{r} \geqslant 1.$$

We can compute the value of $\bar{\pi} \cdot \bar{r}$ for stars grouped according to apparent magnitude * if the luminosity, density, and absorption functions are known. If we introduce the convenient abbreviation

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* For a treatment of the problem when the grouping is according to the intensity of instellar lines, see *D.A.O. Pub.*, **5**, No. 3 ; see also Strömberg, *Mt. Wilson Contr.*, 558, 1936.

$$\psi_m(n) = \int_0^\infty r^n dr \phi(m + 5 - 5 \log r - \alpha[r]) D(r),$$

where ϕ and D are the luminosity and density functions and α is the absorption function, the well-known equations for the mean parallax and mean distance of stars of apparent magnitude m become

$$\bar{\pi}_m = \frac{\psi_m(1)}{\psi_m(2)}, \qquad \bar{r}_m = \frac{\psi_m(3)}{\psi_m(2)}.$$

If we assume that $\alpha = 0$, and put

$$\phi(M) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(M - M_0)^2}{2\sigma^2}}$$

and

$$D(r) = e^{h + k(\log r) + l(\log r)^2}$$

we find that

$$\bar{\pi} \cdot \bar{r} = e^{\frac{(0{\cdot}461\sigma)^2}{1 - {\cdot}08\sigma^2 l}}.$$

This reduces to a formula published by Oort * when $l = 0$.

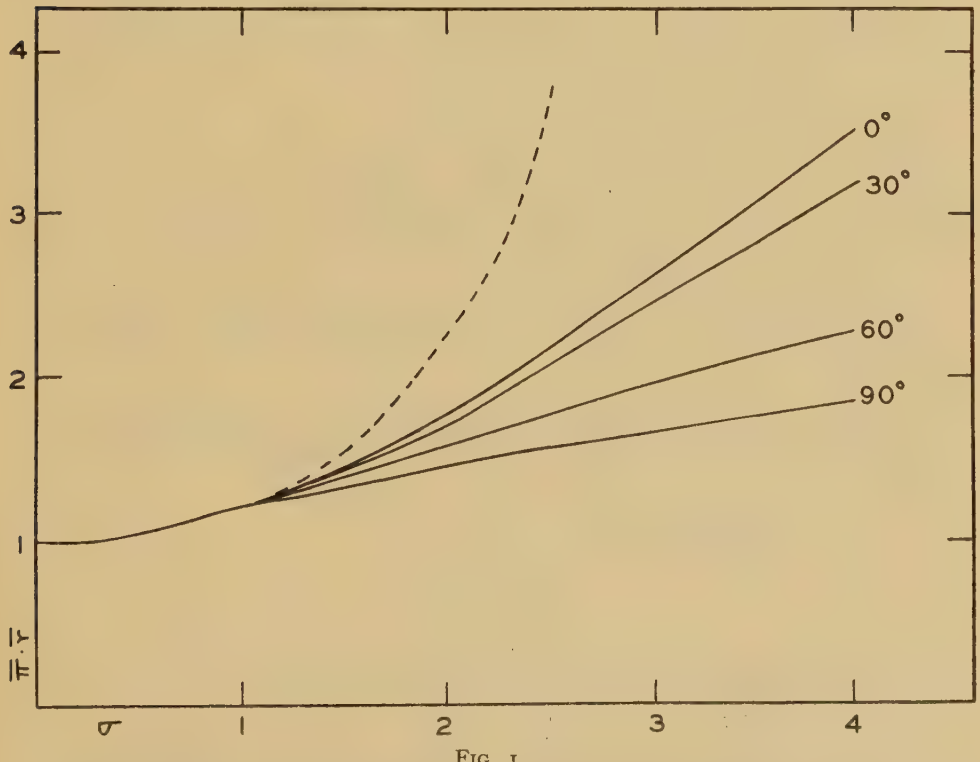

FIG. 1.

Fig. 1 shows $\bar{\pi} \cdot \bar{r}$ as a function of the dispersion in absolute magnitude for different galactic latitudes, the values of l being taken from Table III of

* *B.A.N.*, **132**, 81, 1927.

Kapteyn's paper in *Mt. Wilson Contr.*, 229. The dotted curve is for the case of constant density (or $l=0$). Taking Kapteyn's value of $\sigma=2\cdot5$, we find that the factor $\bar{\pi}\,.\,\bar{r}$ is nearly 30 per cent. larger in the galactic plane than it is at the galactic pole.

Since the mathematical functions adopted for ϕ and D are known not to represent the actual state of affairs in the galaxy, the above results are to be taken only as an indication that one must be on the look-out for changes in $\bar{\pi}\,.\,\bar{r}$ with galactic latitude. Since the density decrease is more rapid in high latitudes than in low, $\bar{\pi}\,.\,\bar{r}$ must be larger in low latitudes than in high.

The existence of an absorption increasing uniformly with distance will cause $\bar{\pi}\,.\,\bar{r}$ to decrease with increasing magnitude, and a density function of the sort given by Bok * causes fluctuations as a function of magnitude amounting to 25 or 30 per cent.

Summary

1. Extension of Milne's work on "Stellar Kinematics and the K-effect" to second order terms shows that, in general, the solar motion as ordinarily defined is a function of the galactic latitude and the distance of the stars to which it is referred. Furthermore, values derived from radial velocities and proper motions in latitude and longitude will not be consistent, in general.

2. Introduction of the special assumptions of galactic rotation does not modify these conclusions. It is shown that the secular parallaxes of distant stars will be made systematically too large by the second order galactic rotation effect. Mean parallaxes derived from tau-components will also be systematically too large, but the effect is much smaller than for secular parallaxes. In both cases the effect is a function of galactic latitude.

3. The influence of irregularities in the distribution of absorbing material in the galaxy is examined.

4. The relationship between mean parallax and mean distance is discussed. It is pointed out that the product $\bar{\pi}\,.\,\bar{r}$ must depend on galactic latitude, and probably varies with apparent magnitude.

I wish to thank Dr. Russell, Dr. Bok, Dr. van de Kamp, and Dr. McCuskey for reading the paper and making important suggestions for its improvement.

Harvard College Observatory :
December 1936.

APPENDIX I

Radial Velocity

$$a_0 = r\left\{\frac{1}{2}\left(\frac{\partial \bar{U}}{\partial X}+\frac{\partial \bar{V}}{\partial Y}\right)_0 \cos^2\beta+\left(\frac{\partial \bar{W}}{\partial Z}\right)_0 \sin^2\beta\right\}$$
$$+r^2\sin\beta\left\{\frac{1}{4}\left(2\frac{\partial^2 \bar{U}}{\partial X\partial Z}+2\frac{\partial^2 \bar{V}}{\partial Y\partial Z}+\frac{\partial^2 \bar{W}}{\partial X^2}+\frac{\partial^2 \bar{W}}{\partial Y^2}\right)_0 \cos^2\beta+\frac{1}{2}\left(\frac{\partial^2 \bar{W}}{\partial Z^2}\right)_0 \sin^2\beta\right\}.$$

* *H.C.*, 371, Table IX, 1931.

$$a_1 = r\left\{\left(\frac{\partial \bar{U}}{\partial Z} + \frac{\partial \bar{W}}{\partial X}\right)_0 \sin\beta\cos\beta\right\}$$
$$+ r^2\cos\beta\left\{\tfrac{1}{2}\left(\frac{\partial^2 \bar{U}}{\partial Z^2} + 2\frac{\partial^2 \bar{W}}{\partial X \partial Z}\right)_0 \sin^2\beta + \tfrac{1}{8}\left(3\frac{\partial^2 \bar{U}}{\partial X^2} + \frac{\partial^2 \bar{U}}{\partial Y^2} + 2\frac{\partial^2 \bar{V}}{\partial X \partial Y}\right)_0 \cos^2\beta\right\}.$$

$$a_2 = r\left\{\tfrac{1}{2}\left(\frac{\partial \bar{U}}{\partial X} - \frac{\partial \bar{V}}{\partial Y}\right)_0 \cos^2\beta\right\}$$
$$+ r^2\left\{\tfrac{1}{4}\left(2\frac{\partial^2 \bar{U}}{\partial X \partial Z} - 2\frac{\partial^2 \bar{V}}{\partial Y \partial Z} + \frac{\partial^2 \bar{W}}{\partial X^2} - \frac{\partial^2 \bar{W}}{\partial Y^2}\right)_0 \sin\beta\cos^2\beta\right\}.$$

$$a_3 = + r^2\left\{\tfrac{1}{8}\left(\frac{\partial^2 \bar{U}}{\partial X^2} - \frac{\partial^2 \bar{U}}{\partial Y^2} - 2\frac{\partial^2 \bar{V}}{\partial X \partial Y}\right)_0 \cos^3\beta\right\}.$$

$$b_1 = r\left\{\left(\frac{\partial \bar{V}}{\partial Z} + \frac{\partial \bar{W}}{\partial Y}\right)_0 \sin\beta\cos\beta\right\}$$
$$+ r^2\cos\beta\left\{\tfrac{1}{2}\left(\frac{\partial^2 \bar{V}}{\partial Z^2} + 2\frac{\partial^2 \bar{W}}{\partial Y \partial Z}\right)_0 \sin^2\beta + \tfrac{1}{8}\left(2\frac{\partial^2 \bar{U}}{\partial X \partial Y} + \frac{\partial^2 \bar{V}}{\partial X^2} + 3\frac{\partial^2 \bar{V}}{\partial Y^2}\right)_0 \cos^2\beta\right\}.$$

$$b_2 = r\left\{\tfrac{1}{2}\left(\frac{\partial \bar{U}}{\partial Y} + \frac{\partial \bar{V}}{\partial X}\right)_0 \cos^2\beta\right\} + r^2\left\{\tfrac{1}{2}\left(\frac{\partial^2 \bar{U}}{\partial Y \partial Z} + \frac{\partial^2 \bar{V}}{\partial X \partial Z} + \frac{\partial^2 \bar{W}}{\partial X \partial Y}\right)_0 \sin\beta\cos^2\beta\right\}.$$

$$b_3 = + r^2\left\{\tfrac{1}{8}\left(2\frac{\partial^2 \bar{U}}{\partial X \partial Y} + \frac{\partial^2 \bar{V}}{\partial X^2} - \frac{\partial^2 \bar{V}}{\partial Y^2}\right)_0 \cos^3\beta\right\}.$$

Tangential Velocity in Longitude

$$a_0 = r\left\{\tfrac{1}{2}\left(-\frac{\partial \bar{U}}{\partial Y} + \frac{\partial \bar{V}}{\partial X}\right)_0 \cos\beta\right\} + r^2\left\{\tfrac{1}{2}\left(-\frac{\partial^2 \bar{U}}{\partial Y \partial Z} + \frac{\partial^2 \bar{V}}{\partial X \partial Z}\right)_0 \sin\beta\cos\beta\right\}.$$

$$a_1 = r\left\{\left(\frac{\partial \bar{V}}{\partial Z}\right)_0 \sin\beta\right\} + r^2\left\{\tfrac{1}{2}\left(\frac{\partial^2 \bar{V}}{\partial Z^2}\right)_0 \sin^2\beta + \tfrac{1}{8}\left(-2\frac{\partial^2 \bar{U}}{\partial X \partial Y} + 3\frac{\partial^2 \bar{V}}{\partial X^2} + \frac{\partial^2 \bar{V}}{\partial Y^2}\right)_0 \cos^2\beta\right\}.$$

$$a_2 = r\left\{\tfrac{1}{2}\left(\frac{\partial \bar{U}}{\partial Y} + \frac{\partial \bar{V}}{\partial X}\right)_0 \cos\beta\right\} + r^2\left\{\tfrac{1}{2}\left(\frac{\partial^2 \bar{U}}{\partial Y \partial Z} + \frac{\partial^2 \bar{V}}{\partial X \partial Z}\right)_0 \sin\beta\cos\beta\right\}.$$

$$a_3 = + r^2\left\{\tfrac{1}{8}\left(2\frac{\partial^2 \bar{U}}{\partial X \partial Y} + \frac{\partial^2 \bar{V}}{\partial X^2} - \frac{\partial^2 \bar{V}}{\partial Y^2}\right)_0 \cos^2\beta\right\}.$$

$$b_1 = r\left\{\left(-\frac{\partial \bar{U}}{\partial Z}\right)_0 \sin\beta\right\}$$
$$+ r^2\left\{\tfrac{1}{2}\left(-\frac{\partial^2 \bar{U}}{\partial Z^2}\right)_0 \sin^2\beta + \tfrac{1}{8}\left(-\frac{\partial^2 \bar{U}}{\partial X^2} - 3\frac{\partial^2 \bar{U}}{\partial Y^2} + 2\frac{\partial^2 \bar{V}}{\partial X \partial Y}\right)_0 \cos^2\beta\right\}.$$

$$b_2 = r\left\{\tfrac{1}{2}\left(-\frac{\partial \bar{U}}{\partial X} + \frac{\partial \bar{V}}{\partial Y}\right)_0 \cos\beta\right\} + r^2\left\{\tfrac{1}{2}\left(-\frac{\partial^2 \bar{U}}{\partial X \partial Z} + \frac{\partial^2 \bar{V}}{\partial Y \partial Z}\right)_0 \sin\beta\cos\beta\right\}.$$

$$b_3 = + r^2\left\{\tfrac{1}{8}\left(-\frac{\partial^2 \bar{U}}{\partial X^2} + \frac{\partial^2 \bar{U}}{\partial Y^2} + 2\frac{\partial^2 \bar{V}}{\partial X \partial Y}\right)_0 \cos^2\beta\right\}.$$

Tangential Velocity in Latitude

$$a_0 = r\left\{\frac{1}{2}\left(-\frac{\partial \bar{U}}{\partial X}-\frac{\partial \bar{V}}{\partial Y}+2\frac{\partial \bar{W}}{\partial Z}\right)_0 \sin\beta\cos\beta\right\}$$
$$+r^2\cos\beta\left\{\frac{1}{2}\left(-\frac{\partial^2 \bar{U}}{\partial X\partial Z}-\frac{\partial^2 \bar{V}}{\partial Y\partial Z}+\frac{\partial^2 \bar{W}}{\partial Z^2}\right)_0 \sin^2\beta+\frac{1}{4}\left(\frac{\partial^2 \bar{W}}{\partial X^2}+\frac{\partial^2 \bar{W}}{\partial Y^2}\right)_0 \cos^2\beta\right\}.$$

$$a_1 = r\left\{\left(-\frac{\partial \bar{U}}{\partial Z}\right)_0 \sin^2\beta+\left(\frac{\partial \bar{W}}{\partial X}\right)_0 \cos^2\beta\right\}$$
$$+r^2\sin\beta\left\{\frac{1}{2}\left(-\frac{\partial^2 \bar{U}}{\partial Z^2}\right)_0 \sin^2\beta+\frac{1}{8}\left(-3\frac{\partial^2 \bar{U}}{\partial X^2}-\frac{\partial^2 \bar{U}}{\partial Y^2}-2\frac{\partial^2 \bar{V}}{\partial X\partial Y}+8\frac{\partial^2 \bar{W}}{\partial X\partial Z}\right)_0 \cos^2\beta\right\}.$$

$$a_2 = r\left\{\frac{1}{2}\left(-\frac{\partial \bar{U}}{\partial X}+\frac{\partial \bar{V}}{\partial Y}\right)_0 \sin\beta\cos\beta\right\}$$
$$+r^2\cos\beta\left\{\frac{1}{2}\left(-\frac{\partial^2 \bar{U}}{\partial X\partial Z}+\frac{\partial^2 \bar{V}}{\partial Y\partial Z}\right)_0 \sin^2\beta+\frac{1}{4}\left(\frac{\partial^2 \bar{W}}{\partial X^2}-\frac{\partial^2 \bar{W}}{\partial Y^2}\right)_0 \cos^2\beta\right\}.$$

$$a_3 = +r^2\left\{\frac{1}{8}\left(-\frac{\partial^2 \bar{U}}{\partial X^2}+\frac{\partial^2 \bar{U}}{\partial Y^2}+2\frac{\partial^2 \bar{V}}{\partial X\partial Y}\right)_0 \sin\beta\cos^2\beta\right\}.$$

$$b_1 = r\left\{\left(-\frac{\partial \bar{V}}{\partial Z}\right)_0 \sin^2\beta+\left(\frac{\partial \bar{W}}{\partial Y}\right)_0 \cos^2\beta\right\}$$
$$+r^2\sin\beta\left\{\frac{1}{2}\left(-\frac{\partial^2 \bar{V}}{\partial Z^2}\right)_0 \sin^2\beta+\frac{1}{8}\left(-2\frac{\partial^2 \bar{U}}{\partial X\partial Y}-\frac{\partial^2 \bar{V}}{\partial X^2}-3\frac{\partial^2 \bar{V}}{\partial Y^2}+8\frac{\partial^2 \bar{W}}{\partial Y\partial Z}\right)_0 \cos^2\beta\right\}$$

$$b_2 = r\left\{\frac{1}{2}\left(-\frac{\partial \bar{U}}{\partial Y}-\frac{\partial \bar{V}}{\partial X}\right)_0 \sin\beta\cos\beta\right\}$$
$$+r^2\cos\beta\left\{\frac{1}{2}\left(-\frac{\partial^2 \bar{U}}{\partial Y\partial Z}-\frac{\partial^2 \bar{V}}{\partial X\partial Z}\right)_0 \sin^2\beta+\frac{1}{2}\left(\frac{\partial^2 \bar{W}}{\partial X\partial Y}\right)_0 \cos^2\beta\right\}.$$

$$b_3 = +r^2\left\{\frac{1}{8}\left(-2\frac{\partial^2 \bar{U}}{\partial X\partial Y}-\frac{\partial^2 \bar{V}}{\partial X^2}+\frac{\partial^2 \bar{V}}{\partial Y^2}\right)_0 \sin\beta\cos^2\beta\right\}.$$

APPENDIX II

Radial Velocity

$$a_0 = r\left\{\frac{1}{2}\left(\frac{1}{R}\frac{\partial \bar{M}}{\partial\theta}+\frac{\partial \bar{L}}{\partial R}+\frac{\bar{L}}{R}\right)_0 \cos^2\beta\right\}.$$

$$a_1 = r^2\left\{\frac{1}{8}\left(-\frac{2}{R}\frac{\partial^2 \bar{M}}{\partial R\partial\theta}-\frac{1}{R^2}\frac{\partial^2 \bar{L}}{\partial\theta^2}+\frac{4}{R^2}\frac{\partial \bar{M}}{\partial\theta}-3\frac{\partial^2 \bar{L}}{\partial R^2}-\frac{3}{R}\frac{\partial \bar{L}}{\partial R}+\frac{3\bar{L}}{R^2}\right)_0 \cos^3\beta\right\}.$$

$$a_2 = r\left\{\frac{1}{2}\left(-\frac{1}{R}\frac{\partial \bar{M}}{\partial\theta}+\frac{\partial \bar{L}}{\partial R}-\frac{\bar{L}}{R}\right)_0 \cos^2\beta\right\}.$$

$$a^3 = r^2\left\{\frac{1}{8}\left(\frac{2}{R}\frac{\partial^2 \bar{M}}{\partial R\partial\theta}+\frac{1}{R^2}\frac{\partial^2 \bar{L}}{\partial\theta^2}-\frac{4}{R^2}\frac{\partial \bar{M}}{\partial\theta}-\frac{\partial^2 \bar{L}}{\partial R^2}+\frac{3}{R}\frac{\partial \bar{L}}{\partial R}-\frac{3\bar{L}}{R^2}\right)_0 \cos^3\beta\right\}.$$

$$b_1 = r^2\left\{\frac{1}{8}\left(\frac{2}{R}\frac{\partial^2\bar{L}}{\partial R\partial\theta}+\frac{3}{R^2}\frac{\partial^2\bar{M}}{\partial\theta^2}+\frac{4}{R^2}\frac{\partial\bar{L}}{\partial\theta}+\frac{\partial^2\bar{M}}{\partial R^2}+\frac{1}{R}\frac{\partial\bar{M}}{\partial R}-\frac{\bar{M}}{R^2}\right)_0\cos^3\beta\right\}.$$

$$b_2 = r\left\{\frac{1}{2}\left(-\frac{1}{R}\frac{\partial\bar{L}}{\partial\theta}-\frac{\partial\bar{M}}{\partial R}+\frac{\bar{M}}{R}\right)_0\cos^2\beta\right\}.$$

$$b_3 = r^2\left\{\frac{1}{8}\left(\frac{2}{R}\frac{\partial^2\bar{L}}{\partial R\partial\theta}-\frac{1}{R^2}\frac{\partial^2\bar{M}}{\partial\theta^2}-\frac{4}{R^2}\frac{\partial\bar{L}}{\partial\theta}+\frac{\partial^2\bar{M}}{\partial R^2}-\frac{3}{R}\frac{\partial\bar{M}}{\partial R}+\frac{3\bar{M}}{R^2}\right)_0\cos^3\beta\right\}.$$

Tangential Velocity in Longitude

$$a_0 = r\left\{\frac{1}{2}\left(\frac{1}{R}\frac{\partial\bar{L}}{\partial\theta}-\frac{\partial\bar{M}}{\partial R}-\frac{\bar{M}}{R}\right)_0\cos\beta\right\}.$$

$$a_1 = r^2\left\{\frac{1}{8}\left(-\frac{2}{R}\frac{\partial^2\bar{L}}{\partial R\partial\theta}+\frac{1}{R^2}\frac{\partial^2\bar{M}}{\partial\theta^2}+\frac{4}{R^2}\frac{\partial\bar{L}}{\partial\theta}+3\frac{\partial^2\bar{M}}{\partial R^2}+\frac{3}{R}\frac{\partial\bar{M}}{\partial R}-\frac{3\bar{M}}{R^2}\right)_0\cos^2\beta\right\}.$$

$$a_2 = r\left\{\frac{1}{2}\left(-\frac{1}{R}\frac{\partial\bar{L}}{\partial\theta}-\frac{\partial\bar{M}}{\partial R}+\frac{\bar{M}}{R}\right)_0\cos\beta\right\}.$$

$$a_3 = r^2\left\{\frac{1}{8}\left(\frac{2}{R}\frac{\partial^2\bar{L}}{\partial R\partial\theta}-\frac{1}{R^2}\frac{\partial^2\bar{M}}{\partial\theta^2}-\frac{4}{R^2}\frac{\partial\bar{L}}{\partial\theta}+\frac{\partial^2\bar{M}}{\partial R^2}-\frac{3}{R}\frac{\partial\bar{M}}{\partial R}+\frac{3\bar{M}}{R^2}\right)_0\cos^2\beta\right\}.$$

$$b_1 = r^2\left\{\frac{1}{8}\left(-\frac{2}{R}\frac{\partial^2\bar{M}}{\partial R\partial\theta}+\frac{3}{R^2}\frac{\partial^2\bar{L}}{\partial\theta^2}-\frac{4}{R^2}\frac{\partial\bar{M}}{\partial\theta}+\frac{\partial^2\bar{L}}{\partial R^2}+\frac{1}{R}\frac{\partial\bar{L}}{\partial R}-\frac{\bar{L}}{R^2}\right)_0\cos^2\beta\right\}.$$

$$b_2 = r\left\{\frac{1}{2}\left(\frac{1}{R}\frac{\partial\bar{M}}{\partial\theta}-\frac{\partial\bar{L}}{\partial R}+\frac{\bar{L}}{R}\right)_0\cos\beta\right\}.$$

$$b_3 = r^2\left\{\frac{1}{8}\left(-\frac{2}{R}\frac{\partial^2\bar{M}}{\partial R\partial\theta}-\frac{1}{R^2}\frac{\partial^2\bar{L}}{\partial\theta^2}+\frac{4}{R^2}\frac{\partial\bar{M}}{\partial\theta}+\frac{\partial^2\bar{L}}{\partial R^2}-\frac{3}{R}\frac{\partial\bar{L}}{\partial R}+\frac{3\bar{L}}{R^2}\right)_0\cos^2\beta\right\}.$$

Tangential Velocity in Latitude

These coefficients are $-\tan\beta$ times the coefficients of the radial velocity.

OCCULTATIONS OF STARS AND SUN BY THE MOON, OBSERVED AT TOMSK UNIVERSITY OBSERVATORY DURING THE YEAR 1936.

N. Goryatscheff.

(ϕ = N. 56° 28′ 06″·3 ; $l = 5^h\ 39^m\ 47^s{\cdot}16$ east of Greenwich)

Date (U.T.)	Local Sidereal Time	Star	Mag.	Phase	Δt
	h m s				s
III 30·711552	11 16 02·0	61 Geminorum	5·9	D	+4·8
IV 25·677247	12 09 00·3	3 Geminorum	5·8	D	+5·9
IV 25·688466	12 25 12·4	4 Geminorum	6·7	D	+3·5
VI 19·202625	4 20 31·4	Sun 2nd cont.	...	D	+3·8
VI 19·204303	4 22 56·7	Sun 3rd cont.	...	R	+6·1
VII 13·829955	21 00 58·5	151 B Arietis	6·7	R	+8·0
VIII 12·865967	23 51 15·1	B.D. +23° 1122	8·6	R	+8·7
VIII 12·881573	0 13 47·1	B.D. +23° 1130	7·5	R	+6·0

Apparent Places of Occulted Stars and Sun

Star	α	δ	Authority
	h m s	° ′ ″	
61 Geminorum	7 23 12·39	+20 23 09·3	Amer. Ephem.
3 Geminorum	6 05 52·38	+23 07 33·7	Amer. Ephem.
4 Geminorum	6 06 38·41	+23 00 41·3	Amer. Ephem.
Sun (2nd c.)	5 50 01·92	+23 25 35·2	Amer. Ephem.
Sun (3rd c.)	5 50 02·34	+23 25 35·3	Amer. Ephem.
151 B Arietis	3 05 42·04	+20 31 12·4	Amer. Ephem.
B.D. +23° 1122	5 53 38·99	+23 08 30·3	Abbadia 2981
B.D. +23° 1130	5 54 33·94	+23 09 04·3	Abbadia 2990

Sun (2nd and 3rd cont.) R = 15′ 46″·1.

Observer : N. Goryatscheff.

Instruments : Zeiss telescope 90-mm. aperture, magnifying power 90. Observations made with eye and ear.

The time-corrections of the chronometer were found from the Bordeaux, Rugby and Nauen wireless signals.

Tomsk, Siberia, U.S.S.R.,
University Observatory :
1937 *March* 2.